Homework 13
Due Tuesday, November 19, at class

1. Consider a sequence of independent coin tosses (i.i.d. Bernoulli random variables)

$$\boldsymbol{x} = [x_1, \ldots, x_N]^T$$

Let $x_n = 1$ denote "heads" and $x_n = 0$ "tails," and $Pr(x_n = 1) = \theta$, $Pr(x_n = 0) = 1-\theta$. The likelihood function is

$$p(\boldsymbol{x}|\theta) \;=\; \theta^{s(\boldsymbol{x})}(1-\theta)^{N-s(\boldsymbol{x})}$$

where $s(\boldsymbol{x}) = \sum_{n=1}^{N} x_n$. Suppose we are interested in estimating θ given $\boldsymbol{x}$. Let's take a Bayesian approach. *A priori* we *believe* that $\theta \approx 1/2$ (*i.e.,* we're tossing a reasonably fair coin), and we know $0 \leq \theta \leq 1$.

a. Show that a beta density prior for θ reflects this prior information. The beta density is given by

$$p(\theta;\alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\theta^{\alpha-1}(1-\theta)^{\alpha-1}$$

where $\alpha \geq 1$ is a shape parameter to be specified by the user and Γ is the Euler gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

b. Show that the beta density is a conjugate prior for the Bernoulli distribution. That is, show that the posterior density also has a beta form.

c. Find the conditional mean estimator $\mathrm{E}[\theta|\boldsymbol{x}]$, a function of α and the data. How does α effect estimator performance? What is the asymptotic (large N) behavior of the estimator?

2. Consider the linear statistical model

$$\boldsymbol{x} = \boldsymbol{H}\boldsymbol{\theta} + \boldsymbol{n}$$

where $\boldsymbol{H}$ is a known $N \times p$ matrix and $\boldsymbol{n} \sim N(\boldsymbol{0}, \boldsymbol{R}_{nn})$. We are interested in estimating $\boldsymbol{\theta}$ given an observation $\boldsymbol{x}$. Let us adopt a Bayesian strategy. Place a multivariate Gaussian prior on the unknown parameter vector: $\boldsymbol{\theta} \sim N(\boldsymbol{0}, \boldsymbol{R}_{\theta\theta})$, independent of $\boldsymbol{n}$. Find the optimal Bayes MSE estimator of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ under squared error loss.