Homework 9

Due Tuesday October 8, at class

Suppose that we have the following signal observation model:

$$\mathbf{x} \;=\; \mathbf{H}\boldsymbol{\theta} \;+\; \boldsymbol{w},$$

where $\mathbf{x} = [x_0, \ldots, x_{N-1}]^T$ is an $N \times 1$ vector of the measured signal samples, $\mathbf{H}$ is a known matrix $N \times N$ with orthonormal columns $\mathbf{h}_0, \ldots, \mathbf{h}_{N-1}$, $\boldsymbol{\theta}$ is an unknown $N \times 1$ parameter vector, and $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$ with σ^2 known. Notice that we can also write the model as

$$\mathbf{x} \;=\; \sum_{n=0}^{N-1} \theta_n \mathbf{h}_n \;+\; \boldsymbol{w}.$$

a. Show that the likelihood function can be expressed as a product of one-dimensional likelihoods:

$$p(\mathbf{x}|\boldsymbol{\theta}) \;=\; \prod_{n=0}^{N-1} p(\mathbf{h}_n^T \mathbf{x}|\theta_n).$$

This shows that $\mathbf{h}_n^T \mathbf{x}$ is a sufficient statistic for θ_n.

b. Suppose that we now want to test for the presence or absence of the n-th component $\theta_n \mathbf{h}_n$. That is, $H_0 : \theta_n = 0$ vs. $H_1 : \theta_n \neq 0$. Express this test as a two-sided composite hypothesis test. (HINT: Recall the test $|t| \overset{H_1}{\underset{H_0}{\gtrless}} \gamma$ from lecture.)

c. Given a specified false alarm probability P_F, give an expression for the threshold γ. Derive an expression for the probability of detection (as a function of θ_n). (HINT: Derive the P_D expression given in lecture for the test $|t| \overset{H_1}{\underset{H_0}{\gtrless}} \gamma$.)

d. Applying this test to each component of the model gives us a decision-theoretic method of "denoising" the observation: retain only the statistically *significant* components of the model and discard the rest. Give an expression for an estimate of the noise-free signal $\mathbf{s} = \mathbf{H}\boldsymbol{\theta}$ using only the components that are detected as present.

e. Put this denoising scheme into action. The *Discrete Cosine Transform* (DCT) is an orthogonal signal transform analogous to the DFT except that the DCT basis functions are real-valued cosines. Matlab has built-in functions `dct` and `idct`, to compute the DCT and inverse DCT, respectively. Since the DCT is an orthogonal transform, it can be expressed as a matrix $\mathbf{H}$ with orthonormal column vectors. Hence, we can build a signal model like the one above where the columns are discrete-time cosines and the parameters $\boldsymbol{\theta}$ are the DCT coefficients. Generate an $N = 256$ length test signal $\mathbf{s}$ by randomly selecting 10 (256-length) DCT basis functions and weighting each by independent, random coefficients $\theta \sim \mathcal{N}(0, 25)$. This synthetic signal then lies in a (random) 10-dimensional subspace of the DCT domain. Add Gaussian white noise with noise power $\sigma^2 = 1$ to the test signal to simulate an observation $\mathbf{x}$. Try out the decision-theoretic signal estimation scheme above for various P_F values and several synthetic signals. Plot and comment on the results.