Toward Network Coding for Interference Networks

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Outline

• Introduction
  – Network coding
  – Wireless models – broadcast erasure networks

• Our system model
  – Finite-field operations
  – Both broadcast and interference constraints

• Upper bound

• Network coding strategy
  – Achieves rates asymptotically close to u.b.

• Capacity gains due to fading
Network Coding – Wireline Networks

- All links at rate 1
- Single-source multicast
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Network Coding – Wireline Networks

• All links at rate 1
• Single-source multicast
• Upper bound on rate $R$ for each destination is 2
  – Same as min-cut
• Routing cannot achieve this
  – If $? = b$, $D_2$ only receives $b$
• Network coding can
  – By coding at intermediate node

![Diagram of network coding](image)
**Network Coding – General Wireline Networks**

Theorem [AhCaLiYe00]: Multicast capacity $= \min_i \{ \text{min-cut for } D_i \}$

- **Upper bound**
  - Min-cut bound for each $D_i$

- **Achievability** [Ho et al 03]
  - Source sends messages from $F_q$
  - Nodes perform Random Linear Coding (RLC) over received messages:
    $$ C = \alpha_1 a + \alpha_2 b, \quad \alpha_i \in F_q $$
  - $D_1$ decodes source messages from received vectors: $Y_1=(Y_{11}=a, Y_{12}=C)$
  - Achieves rates arbitrarily close to min-cut bound for sufficiently large $q$
Broadcast Erasure Networks (BEN)

- Directed graph $G=(V,E)$
- Each link $e \in E$ is independent erasure channel
  - $P(Y_{14} \mid X_1, X_2) = P(Y_{14} \mid X_1)$
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- Broadcast constraint
  - $v_1$ must send same $X_1$ along both $(v_1, v_3)$ and $(v_1, v_4)$
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  - $X_4 = f(Y_{14}, Y_{24})$
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Does not model interference
Broadcast Erasure Networks -- Capacity

- Results for directed acyclic graphs
- **Theorem** [DanGow04, LunMed04]:
  \[ \text{Capacity} = \min_i \{ \text{generalized min-cut for } D_i \} \]
  - e.g., \((1 - \varepsilon_{13}\varepsilon_{14}) + (1 - \varepsilon_{24})\)
- **Upper bound**
  - Follows from min-cut bound
    \[ \min_{\text{cut}} I(X_{\text{cut-l}}; Y_{\text{cut-r}} | X_{\text{cut-r}}) \]
- **Achievability**
  - [DanGow04] Random coding at nodes
    - need to keep track of erasure patterns
  - [LunMed04] model as Hypergraph, RLC at nodes
    - track flow of innovative packets
    - generalized to arbitrary arrival processes and correlated erasure patterns
Wireless Broadcast and Interference Networks (WBAIN)

• Above and other results model broadcast but not interference
• Interference is challenging to analyze
• Capacity region not known for even simple network configurations

− Single-relay channel

− Interference channel

− …
WBAIN – A Finite-Field Model

- Directed acyclic graph $G=(V,E)$
- **Broadcast** constraint
  - $v_1$ must send **same** $X_1$ along both $(v_1, v_3)$ and $(v_1, v_4)$
WBAIN – A Finite-Field Model

- Directed acyclic graph \( G = (V,E) \)
- Broadcast constraint
  - \( v_1 \) must send same \( X_1 \) along both \((v_1, v_3)\) and \((v_1, v_4)\)
- Model power constraint by rate
  - \( v_i \) can send at rate \( \leq R_i \)
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- All operations over finite field $F_q$
  - Each node transmits vectors from $F_q$
  - $\log q \geq \max_i R_i$
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- Two reception models
  - With or without fading
**WBAIN – A Finite-Field Model (contd.)**

- **Non-fading model:**
  \[ v_4 \text{ receives } Y_4, \text{ where } Y_4 \]
  - \[ = X_1 + X_2 \in F_q \text{ with prob. } 1 - \varepsilon_4 \]
  - \[ = \phi \text{ (erasure) with prob. } \varepsilon_4 \]
  - Erasures are independent across receivers

- **Fading model:**
  As above, except when no erasure
  - \[ Y_4 = h_{14} X_1 + h_{24} X_2 \]
  - \[ h_{ij} \text{ uniform i.i.d. over } F_q \]
Upper Bound on WBAIN with Fading

• Bound on capacity of finite-field MAC

\[
C_3(q) = \max_{\{H(X_j) \leq R_j\}} I(Y_3; X_1, X_2 | h_{13}, h_{23})
\]

\[
= \max_{\{H(X_j) \leq R_j\}} H(Y_3 | h_{13}, h_{23}) - H(Y_3 | X_1, X_2, h_{13}, h_{23})
\]

\[
= \max_{\{H(X_j) \leq R_j\}} H(h_{13}X_1 + h_{23}X_2 | h_{13}, h_{23}) - 0
\]

\[
= q^{-2}((q-1)(R_1 + R_2) + (q-1)^2 \min \{R_1+R_2, \log q\})
\]

\[
\leq \min \{(1-q^{-1})(R_1 + R_2), (1-q^{-2}) \log q\}
\]
**Upper Bound on WBAIN with Fading**

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  \[ = \max_{\{H(X_j) \leq R_j\}} H(Y_3 | h_{13}, h_{23}) - H(Y_3 | X_1, X_2, h_{13}, h_{23}) \]
  \[ = \max_{\{H(X_j) \leq R_j\}} H(h_{13}X_1 + h_{23}X_2 | h_{13}, h_{23}) - 0 \]
  \[ = q^{-2}((q-1)(R_1 + R_2) + (q-1)^2 \min \{R_1+R_2, \log q\}) \]
  \[ \leq \min \{(1-q^{-1})(R_1 + R_2), (1-q^{-2}) \log q\} \]

- Node i receives transmissions from nodes in J
  \[ C_i(q) \leq \min \{(1-q^{-1})(\sum_{j \in J} R_j), (1-q^{-\delta_i}) \log q\} \]
Upper Bound on WBAIN with Fading

• Bound on capacity of finite-field MAC

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\[ = \max \{H(X_j) \leq R_j\} H(h_{13}X_1 + h_{23}X_2 | h_{13}, h_{23}) - 0 \]

\[ = q^{-2}((q-1)(R_1 + R_2) + (q-1)^2 \min \{R_1+R_2, \log q\}) \]

\[ \leq \min \{(1-q^{-1})(R_1 + R_2), (1-q^{-2}) \log q\} \]

• Node i receives transmissions from nodes in J

\[ C_i(q) \leq \min \{(1-q^{-1})\sum_{j \in J} R_j, (1-q^{-\delta(i)}) \log q\} \]

• More generally, total rate across cut U bounded by

\[ C_U(q) \leq \max \{H(X_j) \leq R_j\} I(Y_3, Y_4; X_1, X_2 | H_{1,2;3,4}) \]

\[ \leq \min \{\sum_j (1-q^{-\delta_0(j)}) R_j, \log q \sum_i 1-q^{-\delta(i)}\} \]
Upper Bound (contd.)

- Consider Broadcast Erasure Network $T(G)$ having same topology and rates as $G$, with
  - no interference, e.g., $Y_4$ receives $(X_1, X_2)$
Upper Bound (contd.)

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  - no interference, e.g., $Y_4$ receives $(X_1, X_2)$
  - each broadcast link has independent erasures with probability $q^{-1}$, e.g., (1,3), (1,4), (2,4)
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  - MAC sum-rate constraint through aux. edge rate constraint, $R_i = (1-q^{-\delta_l(i)})\log q$
**Upper Bound (contd.)**

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- **Theorem:** Capacity of WBAIN $G$ over $\mathbb{F}_q$, $C_q$
  \[ \leq \text{Capacity of BEN, } T(G), C_s(q) \]
  \[ = \min_i \{\text{generalized min-cut for } D_i\} \]
Coding Strategy for WBAIN with Fading

- For any $\delta > 0$, there exists a flow vector $\{f_p\}$ for all paths $\{p\}$ between s-d in BEN T(G) such that
  - $\sum_p f_p = C_s \cdot (1 - \delta)$
  - $\sum_{p: v_i \in p} \frac{f_p}{(1 - \varepsilon_h(v_i, p))} \leq R_i \cdot (1 - \delta)$
  - $h(v_i, p)$ is the next hop from $i$ on path $p$
### Coding Strategy for WBAIN with Fading

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  - $\sum_p f_p = C_s \cdot (1-\delta)$
  - $\sum_{p: v_i \in p} f_p / (1 - \varepsilon_{h(v_i,p)}) \leq R_i \cdot (1-\delta)$
    - $h(v_i,p)$ is the next hop from $i$ on path $p$
- Coding strategy:
  - Source $s$ gets messages at rate $C_s \cdot (1-\delta)$
**Coding Strategy for WBAIN with Fading**

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- Coding strategy:
  - Source $s$ gets messages at rate $C_s \cdot (1 - \delta)$
  - $s$ injects RLC of received messages at rate $\sum_p f_p / (1 - \epsilon_{h(s,p)})$
**Coding Strategy for WBAIN with Fading**

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  - Node $v_i$ injects RLC of received messages at rate $\sum_{p: v_i \in p} f_p / (1-\epsilon_{h(v_i,p)})$
Coding Strategy for WBAIN with Fading

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  - $\sum_p f_p = C_s \cdot (1-\delta)$
  - $\sum_{p: v_i \in p} f_p \cdot (1-\epsilon_{h(v_i,p)}) \leq R_i \cdot (1-\delta)$
    • $h(v_i,p)$ is next hop from i on path p

• Coding strategy:
  - Source s gets messages at rate $C_s \cdot (1-\delta)$
  - s injects RLC of received messages at rate $\sum_p f_p \cdot (1-\epsilon_{h(s,p)})$
  - Node $v_i$ injects RLC of received messages at rate $\sum_{p: v_i \in p} f_p \cdot (1-\epsilon_{h(v_i,p)})$

• Theorem: $C_s \cdot (1-O(1/q)) \cdot (1-\delta)$ is achievable in G with uniform i.i.d. fading
Coding Strategy for WBAIN with Fading (contd)

Main steps in achievability proof:

- Track the flow of innovative packets
- Fading helps to maintain innovation rates over different links in a cut
  - in spite of broadcast and interference
- “Bad” fading at node $v_j$ -- $h_j = (h_{ij})_{i} = 0$ or dependent on $\{h_k\}$ -- reduces rate of innovation by at most $(1-O(1/q))$
- At each hop of path $p$ the rate of innovation is at least $g_p = f_p \cdot (1-O(N_o/q))$
  - $N_o = $ diameter of $G$
- Achieved rate $= \sum_p g_p = C_s \cdot (1-O(N_o/q)) \cdot (1-\delta)$
Tight bounds on Capacity of WBAIN with fading

Theorem: \( C_s \cdot (1 - O(1/q)) \leq C_q \leq C_s \)

- Also holds for heterogenous networks having both wireless and wireline links:
  - Each node can have both types of incoming and outgoing links
  - Node receives weighted sum of vectors sent over incident wireless links, \( Y_4 = h_{14}X_1 + h_{24}X_2 \)
  - Node receives separate information over incoming wireline links, \( Y_7 = (X_5, X_6) \)
  - Similarly, when node transmits
Capacity Gains due to Fading – An Example

- Heterogenous network: wireless at cut $U$, wireline otherwise
- $R_1$ and $q$ s.t. $U$ is bottleneck cut
  - e.g., $R_1 = \log q$
- Upper bound:
  \[ C_s \approx \sum_{i=1}^{5} R_1 (1-\varepsilon_i) = R_1 (5-\sum_i \varepsilon_i) \]
- Fading: our strategy achieves
  \[ C_s \cdot (1-O(1/q))(1-\delta) \]
- No fading: capacity is bounded by
  \[ R_1 (1-\prod_i \varepsilon_i) \]
- \(~5\)-fold increase in capacity with fading
  - Higher for graphs with larger bottleneck cut
Summary and Future Work

• Finite-field model of interference networks
  – All operations over a finite field
  – Incorporates both broadcast and interference constraints
  – Allows for fading

• Asymptotically tight bounds on capacity for uniform iid fading
  – Upper bound based on results for Broadcast Erasure Networks
  – Achievability through network coding

Some Interesting Issues

• Non-uniform fading?

• Achievable rates under no fading?

• What can we infer about Gaussian channels?
  – Limit of finite-field channels under appropriate distribution remapping?