Exchange Algorithms for the Design of Linear Phase FIR Filters and Differentiators Having Flat Monotonic Passbands and Equiripple Stopbands *

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Abstract

This paper describes a modification of a technique proposed by Vaidyanathan for the design of filters having flat passbands and equiripple stopbands. The modification ensures that the passband is monotonic and does so without the use of concavity constraints. Another modification described in this paper adapts the method of Vaidyanathan to the design of lowpass differentiators having a specified degree of tangency at $\omega = 0$.

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1 Introduction

Linear phase FIR filters having very flat passbands and equiripple stopbands are important for several applications [21]. For example, consider the removal of high frequency noise from a low frequency signal by lowpass filtering: To reduce the distortion of the signal introduced by the filter, the use of a filter having a very flat passband is desirable. To maximize the stopband attenuation, the use of a filter having an equiripple stopband is desirable. Filters having very flat passbands are also useful in applications in which a filter appears in cascade with other filters, such as in a long-distance communication channel with "repeater stations" [3]. Furthermore, in [17] Steffen shows that such filters have good approximation properties. Briefly, filters that achieve a specified degree of flatness at $\omega = 0$ preserve the moments of an input signal to a specified degree, see also [14].

Figure 1 shows the frequency response amplitude of a length 33 filter having these characteristics. It equals 1 at $\omega = 0$ and has 21 derivatives equal to 0 at $\omega = 0$. The stopband edge, ω_s, and the Chebyshev error in the stopband, δ_s, are shown in the figure. The parameters used in this paper are:

N	filter length
L	degree of flatness at $\omega = 0$
ω_s	stopband edge
δ_s	stopband Chebyshev error.

By degree of flatness we mean the number of derivatives of the frequency response amplitude equal to 0, or equivalently, the number of vanishing moments of the impulse response equal to zero.

The design of linear phase FIR filters having very flat passbands and equiripple stopbands has been studied by several authors. Darlington [2] described some transformation principles for filters of this type. Kaiser, Steiglitz and Parks have used linear programming methods [7, 18, 19]. While linear programming is a very general and flexible design method for filter design, it is more computationally intensive and is no more immune to numerical difficulties than are exchange algorithms. In [21] Vaidyanathan presents a method based upon the Remez exchange algorithm. The method he describes employs the Parks-McClellan algorithm [9] and a special filter structure. This structure, which has also been used by Schüßler and Steffen [13, 14, 17], enforces a specified degree of flatness at $\omega = 0$ and results in a design algorithm with good numerical properties and a filter implementation with good sensitivity properties. However, the filters obtained by the method described in [21] do not necessarily have monotonic passbands, which is sometimes a requirement.

This short paper describes a modification to the filter design method of [21]. It produces odd-length linear phase filters with equiripple stopbands and *monotonic* passbands having a specified degree of flatness at $\omega = 0$. Although passband monotonicity can be ensured by linear programming methods (by the use of derivative constraints), it is preferable to use the modification of [21] described below, because of its (1) computational efficiency, (2) good numerical properties during the design, and (3) its low sensitivity filter structure. This paper also discusses bandpass filter design and adapts the techniques of [21] to the design of lowpass differentiators.

2 The Design Algorithm

The structure described in [21], shown in fig 2, achieves an L degree of flatness at $\omega = 0$. The filter length N must be odd, otherwise the delay component is a fractional delay which we wish to avoid. The filter transfer function is given by

$$H(z) = z^{-(N-1)/2} + H_1(z)H_2(z) \tag{1}$$

where $H_1(z)$ is given by

$$H_1(z) = \left(\frac{1-z^{-1}}{2}\right)^L. \tag{2}$$

Taking H_2 to be a highpass filter whose impulse response is symmetric and of length $N-L$, $H_2(e^{j\omega})$ can be written [9] as $H_2(e^{j\omega}) = e^{-j\frac{N-L-1}{2}\omega}A_2(\omega)$ where $A_2(\omega)$ is the frequency response amplitude, a real valued function of ω. When L is chosen to be even, $\left(\frac{1-e^{-j\omega}}{2}\right)^L$ can be written as

$$\left(\frac{1-e^{-j\omega}}{2}\right)^L = \left(e^{-j\omega/2}\right)^L (-1)^{L/2}\left(\sin\frac{\omega}{2}\right)^L$$

where we have used the identity $\frac{1-e^{-j\omega}}{2} = je^{-j\omega/2}\sin\frac{w}{2}$. Therefore $H(e^{j\omega})$ can be written as $H(e^{j\omega}) = e^{-j\frac{N-1}{2}\omega}A(\omega)$ where the frequency response amplitude can be written as

$$A(\omega) = 1 + A_2(\omega)(-1)^{L/2}\left(\sin\frac{\omega}{2}\right)^L. \tag{3}$$

It should be noted that the use of L, here and below, includes all the derivatives that are made to match the desired response, and so includes the 0^{th} derivative.

Let $M = (N-L-1)/2$ and denote the filter coefficients of H_2 by $h_2(0), \ldots, h_2(N-L-1)$. $A_2(\omega)$ can be written as

$$A_2(\omega) = h_2(M) + 2\sum_{n=0}^{M-1} h_2(n)\cos(\omega(M-n)). \tag{4}$$

Two approaches to the problem formulation for which exchange algorithms can be used are the following:

1. Specify N, L, ω_s; minimize δ_s.

2. Specify N, L, δ_s; minimize the passband width, (minimize ω_s).

The first of these two options is the traditional approach in which the bands of interest are well defined and the Chebyshev norm of the error function over those bands is minimized. The second version is a variation of this approach in which the Chebyshev error in the stopband is specified but the band edge, however, is not fixed. In this case, no band edge is actually used during the course of the design procedure. The band edge that results is the one that corresponds to the specified Chebyshev error δ_s and the specified degree of flatness L.

2.1 Specifying ω_s

The first of these two approaches is solved by applying the Remez exchange algorithm [9] over just the stopband. In other words, the modification made to the method of [21] is to simply weight the passband error by 0. The use of the Remez algorithm in this way will yield the coefficients of H_2 that minimize

$$\left\| 1 + A_2(\omega)(-1)^{L/2} \left(\sin\frac{\omega}{2}\right)^L \right\|_\infty \tag{5}$$

over the stopband. On each iteration, a reference set of stopband frequencies is updated and the filter H_2 is found such that $A(\omega)$ alternately interpolates δ_s and $-\delta_s$ over the reference set frequencies. The size of the reference set is $q = \frac{N-L+3}{2}$. Let $\omega_1, \ldots, \omega_q$ denote the reference set frequencies ordered in increasing order. The equation

$$A(\omega_i) = \delta_s(-1)^i \tag{6}$$

that appears in the course of the Remez algorithm becomes

$$A_2(\omega_i) = \frac{\delta_s(-1)^i - 1}{(-1)^{L/2}\left(\sin\frac{\omega_i}{2}\right)^L} \tag{7}$$

which is linear in the coefficients of H_2 and δ_s. Solving these equations for $1 \leq i \leq q$ gives the coefficients of H_2 and δ_s. These can be found efficiently by using the interpolation formulas as in the Park-McClellan algorithm. The reference set is updated as in the Parks-McClellan algorithm and a new filter H_2 is found, and so on, until convergence is obtained. Quadratic convergence to the unique optimal solution is guaranteed by the appropriate use of the Remez algorithm.

It should be noted that any implementation of the Remez algorithm which allows the user to give an arbitrary weighting function and an arbitrary desired magnitude response can be used:

Setting

$$W(\omega) = \begin{cases} 0 & \text{for } \omega < \omega_s \\ (-1)^{L/2} \left(\sin \frac{\omega}{2}\right)^L & \text{for } \omega \geq \omega_s \end{cases} \tag{8}$$

and

$$D(\omega) = \frac{-(-1)^{L/2}}{\left(\sin \frac{\omega}{2}\right)^L}, \tag{9}$$

Eq (5) becomes

$$||(A_2(\omega) - D(\omega))W(\omega)||_\infty \tag{10}$$

which is the appropriate formulation for use with the Remez algorithm.

2.2 Specifying δ_s

To specify δ_s and leave the stopband edge variable, we use an approach similar to that of [15]. Like the Remez algorithm, this approach employs a set of stopband reference frequencies. On each iteration (1) an interpolation problem is solved and (2) the reference set is updated. The reference set here, however, does not contain the stopband edge (indeed, it is not specified). Therefore the reference set contains $\frac{N-L+1}{2}$ stopband frequencies.

Given a set of reference frequencies, the filter that alternately interpolates $-\delta_s$ and δ_s is found. The interpolation is such that the filter interpolates $-\delta_s$ at the first reference set frequency. Note that because δ_s is specified by the user, it does not have to be found as in the Remez algorithm. Also note that a filter can be found that satisfies this interpolation requirements because the number of reference set frequencies equals the number of filter parameters. At each iteration, the local extremal frequencies of $A(\omega)$ in $(0, \pi]$ are found and are taken to be the reference set frequencies for the next iteration.

In fig 3, the circular marks indicate the reference set frequencies upon convergence when this approach is used. Notice that the stopband edge is not included among these circular marks and its location is controlled only indirectly. However, with this approach, the user can directly specify the δ_s parameter.

This approach produces the same set of filters as does the use of the Remez algorithm described above, however, it gives a different way to specify the filter parameters in the design process. Note that this approach is also similar to the approach of Hofstetter, Oppenheim and Siegel to the design of extra-ripple filters [5, 6]. The similarity lies in (1) the ability to specify δ and the use of this specified value during the interpolation process and (2) the variability of the band edge. The approach described above is also like the algorithm of Hofstetter et al in that, while we have no proof of convergence, in practice the algorithm duplicates the rapid, robust convergence of the Remez algorithm.

2.3 Passband Monotonicity

The passband can be shown to be monotonic by the following reasoning. Recall that when no degree of flatness is imposed upon $A(\omega)$ the maximum number of frequencies in $[0, \pi]$ at which the derivative of $A(\omega)$ equals zero is $(N+1)/2$ [11]. Note also that additional degrees of flatness imposed at $\omega = 0$ reduces the number of frequencies at which $A'(\omega)$ can equal zero. Because the filters produced by the methods described above have the property that $A'(\omega)$ equals zero at $(N+1-L)/2$ frequencies in the stopband, it appears that there can be no passband frequencies (other than $\omega = 0$) at which $A'(\omega)$ equals zero. Therefore, the passband will be monotonic.

3 Bandpass Filter Design

This method can also be applied to the design of bandpass filters having very flat passbands. In this case, we specify a passband frequency, ω_p, where we wish to impose flatness constraints. Fig 4 shows the frequency response amplitude of a length 55 bandpass filter. The deviations from 0 in the first and second stopbands are denoted by δ_1 and δ_2 respectively.

The appropriate filter structure has the transfer function $H(z) = z^{-(N-1)/2} + H_1(z)H_2(z)$ with

$$H_1(z) = \left(\frac{1 - 2(\cos\omega_p)z^{-1} + z^{-2}}{4}\right)^{L/2} \tag{11}$$

where L is even, N is odd, and H_2 is a filter whose impulse response is symmetric and of length $N - L$. The overall frequency response $H(e^{j\omega})$ can then be written as $H(e^{j\omega}) = e^{-j\frac{N-1}{2}\omega}A(\omega)$ where the frequency response amplitude $A(\omega)$ is given by

$$A(\omega) = 1 + (-1)^{L/2}\left(\frac{\cos\omega_p - \cos\omega}{2}\right)^{L/2} A_2(\omega) \tag{12}$$

and where $A_2(\omega)$ is given by Eq (4).

In keeping with the previous discussion, we desire that the passband be monotonic on both sides of ω_p. To ensure this behavior in the exchange algorithms described below, it is necessary that L be a multiple of 4. When 4 divides L, the zeros of $H_1(z)$ have even multiplicity, making $A_1(\omega)$ a nonnegative function. Then $A(\omega)$ is concave over the passband with appropriately chosen H_2.

As above, there are two approaches for which simple exchange algorithms are well suited:

1. Specify N, L, ω_p, stopband edges, $K = \delta_2/\delta_1$; minimize δ_1.

2. Specify N, L, ω_p, δ_1, δ_2; minimize passband width.

First we describe approach (1), which uses the Remez algorithm with a zero-weighted passband. Because our approach places all the reference set frequencies in the stopbands, and because the Remez algorithm requires that the error function alternate sign over the reference frequencies, the reference set must contain exactly one stopband edge at each iteration. For example, see fig 4 in which the circular marks indicate the reference frequencies upon convergence. In this figure, the first stopband edge is included in the final reference set, but the second is not. Note that bandpass filters designed such that the passband is concave and flat at ω_p have passbands that are generally quite symmetric around ω_p. For this reason, we suggest that the stopband edges are taken to be $\omega_a = \omega_p - \omega_t$ and $\omega_b = \omega_p + \omega_t$.

The reference set is updated by the following procedure: First compute the set of all local extremal frequencies of $A(\omega)$ in $[0, \pi]$. Calling this set R, remove ω_p from R. R will then contain either $\frac{N-L+1}{2}$ or $\frac{N-L+3}{2}$ frequencies. If R contains $\frac{N-L+3}{2}$ frequencies, then remove either 0 or π as follows: if $|A(\pi)| < K|A(0)|$ then remove π, otherwise remove 0. Next, append either ω_a or ω_b to R: if $|A(\omega_b)| < K|A(\omega_a)|$ then append ω_a, otherwise append ω_b. R is the new reference set and has size $\frac{N-L+3}{2}$.

On each iteration, the filter H_2 is found such that $A(\omega)$ interpolates $\delta_1(-1)^i$ over the reference set frequencies in the first stopband and $K\delta_1(-1)^i$ in the second stopband. The resulting interpolation equations are linear in the coefficients of H_2 and δ_1. Convergence to a filter with a concave passband is quadratic.

A similar algorithm is used for approach (2) in which δ_1 and δ_2 are specified and the stopband edges are left variable. The reference set is updated in the same manner, except no stopband edge is appended to R. Let $\omega_1, \ldots \omega_q$ denote the reference set frequencies ordered in increasing order. On each iteration, the filter H_2 is found such that

$$A(\omega_i) = (-1)^{i+c}\delta_1 \qquad \text{for } \omega_i < \omega_p \tag{13}$$

$$A(\omega_i) = (-1)^{i+c+1}\delta_2 \quad \text{for } \omega_i > \omega_p \tag{14}$$

where c equals 0 or 1, whichever gives $A(\omega) = -\delta_1$ at the highest reference frequency less than ω_p. If the filter in fig 4 were designed by specifying δ_1 and δ_2, the reference set upon convergence would exclude the first stopband edge $\omega_a = \omega_p - \omega_t$.

4 Lowpass Differentiators

Lowpass digital differentiators can also be designed using the approach described above. However, the parameterization for differentiators having a specified degree of tangency at $\omega = 0$ is more complicated, and the simple structure used above must be modified. Let $C(\omega) = 1 - \cos\omega$ and let L and N both be even. The frequency response amplitude of a length N differentiator with L degrees of flatness can be expressed as

$$A(\omega) = \left(\sin\frac{\omega}{2}\right)\left[2 + C(\omega)\left[d_1 + C(\omega)\left[d_2 + C(\omega)\left[\cdots + C(\omega)\left[d_{L/2-1} + C(\omega)A_2(\omega)\right]\right]\right]\right]\right] \quad (15)$$

where $A_2(\omega)$ is an arbitrary cosine polynomial of degree $(N-L-1)/2$ and the first few d_i are as follows. $d_1 = 1/6$, $d_2 = 3/80$, $d_3 = 5/448$, $d_4 = 35/9216$, $d_5 = 63/45056$, $d_6 = 231/425984$. The general formula appears to be given by

$$d_k = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!\ \cdot(2k+1)\cdot 2^{2k-1}}. \quad (16)$$

For the amplitude given in Eq (15), $A(0) = 0$, $A'(0) = 1$, and $A^{(k)}(0) = 0$ for $k = 2, \ldots, L$.

Because the method described above for lowpass filter design uses a reference set of *stopband* frequencies only, exactly the same procedure can be used here. Accordingly, it is possible to either (1) specify the stopband edge ω_s and leave δ_s variable or (2) specify δ_s and leave ω_s variable. As above, the interpolation equations are linear in the coefficients of $A_2(\omega)$ and δ_s.

Figure 5 shows the filter structure of an even-length differentiator for which $L = 6$. In the figure, $H(z)$ is a linear-phase transfer function of order $N-L-1$. A maximally-flat differentiator can be obtained by setting $H(z) = 0$, but see also [8, 14]. The structure for odd-length differentiators is similar. Figure 6 shows the frequency response amplitude of a length 58 digital differentiator designed by this approach.

For odd length differentiators, the amplitude response can be written as

$$A(\omega) = (\sin\omega)\left[1 + C(\omega)\left[d_1 + C(\omega)\left[d_2 + C(\omega)\left[\cdots + C(\omega)\left[d_{L/2-1} + C(\omega)A_2(\omega)\right]\right]\right]\right]\right] \quad (17)$$

where $A_2(\omega)$ is an arbitrary cosine polynomial and the first few d_i are as follows. $d_1 = 1/3$, $d_2 = 2/15$, $d_3 = 2/35$, $d_4 = 8/315$, $d_5 = 8/693$. The general formula appears to be

$$d_k = \frac{k!}{1\cdot 3\cdot 5\cdots(2k+1)}.$$

For the amplitude given in Eq (17), $A(0) = 0$, $A'(0) = 1$, and $A^{(k)}(0) = 0$ for $k = 2, \ldots, L$.

5 Discussion and Conclusion

Even-though the use of structures improves the numerical properties of the design procedure, there are limits on the filter length due to the difficulty associated with $(\sin \omega/2)^L$ in Eq (7) when L is very large. The reader is referred to [21] for discussions on IFIR filter design and implementation considerations, including coefficient sensitivity and roundoff noise.

It should be noted that passband monotonicity is achieved by this method without any explicit constraints on the concavity of the frequency response amplitude. It is obtained simply by locating all reference set frequencies in the stopband, so that the passband is shaped by the constraints embedded in the filter structure. Consequently, in the design of lowpass filters by the algorithm described above, the location of the passband edge arises as the result of the stopband specification, which is given in terms of either ω_s or δ_s, but not both. Similar tradeoffs between the ability to directly achieve specified parameters are discussed for equiripple lowpass and bandpass filters in [15].

Note that when L is taken to be 2 for the lowpass case, the filter can be expressed analytically using Chebyshev polynomials [11]. More interestingly, if L is taken to be 4 for the bandpass case, then the subset of maximal ripple bandpass filters can be found using analytic methods involving Zolotarev polynomials as described by Chen and Parks in [1]. Analytic solutions for higher values of L in each case do not appear to be known.

We also wish to note that filters minimizing an integral square error having a specified degree of flatness at $\omega = 0$ are discussed in [10, 13, 14].

The lowpass filters designed by the method described above are analogous to the classical type II Chebyshev (or inverse Chebyshev) IIR filters [9]. Filters analogous to the classical type I Chebyshev IIR filters can be designed by the same principles. Linear phase FIR filters that are analogous to the classical Butterworth and Elliptic IIR filters are the maximally flat FIR filters of [4] and the equiripple FIR filters obtained by the Parks-McClellan algorithm. Thus, FIR analogues to each of the four classical IIR filter types can be designed without the use of general linear programming methods and without the need to explicitly impose linear constraints. The advantages of this is that (1) linear programming methods can be computationally intensive and (2) the use of linear constraints on derivatives become ill-conditioned for modest filter lengths. It is interesting to note (1) that maximally flat filters can be designed by employing simple filter structures [20], (2) that equiripple filters can be designed by employing exchange algorithms, and (3) that the filters described in this paper can be designed by combining the use of a simple structure and by employing exchange algorithms. This is satisfying because the characteristics of the filters designed in this paper combine the characteristics of maximally flat filters and equiripple filters.

Also, recall that the four classical IIR filter types all have an equal number of poles and zeros. It is possible to design IIR filter with an unequal number of poles and zeros by combining the techniques described above with the rational Remez exchange algorithm discussed in [16].

Matlab programs are available from the authors or electronically on the World Wide Web at URL *http://www-dsp.rice.edu*.

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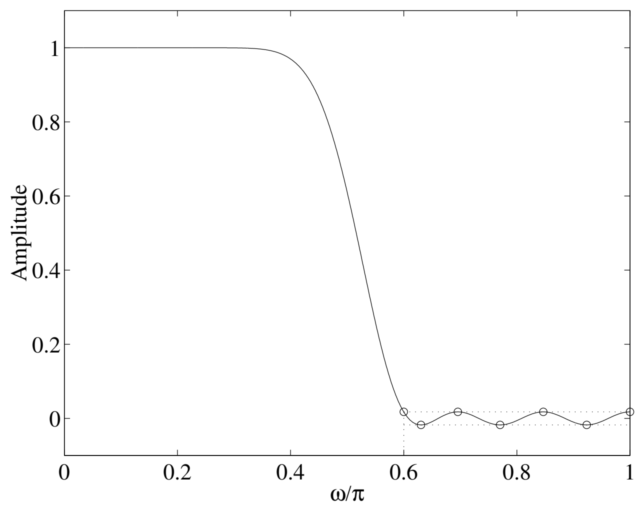

Figure 1: $N = 33$, $L = 22$, $\omega_s = 0.6\pi$, $\delta_s = 0.0175$

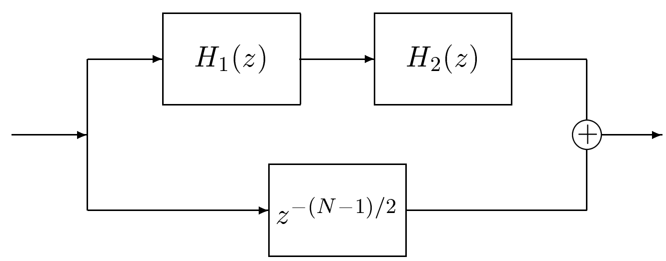

Figure 2: Filter structure for implementation and design of lowpass filter.

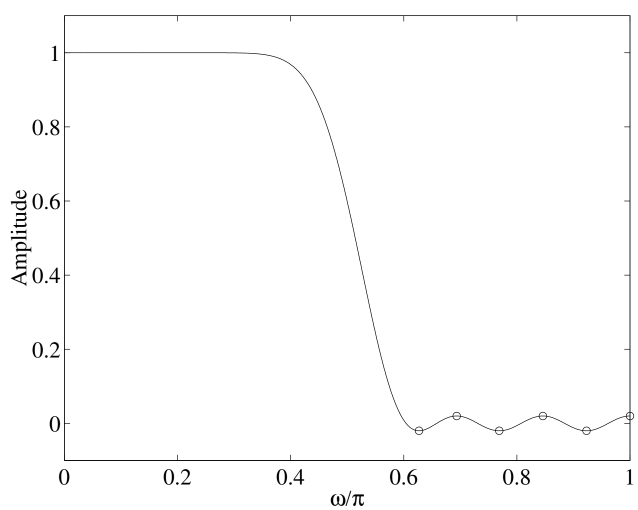

Figure 3: $N = 33$, $L = 22$, $\delta_s = 0.02$

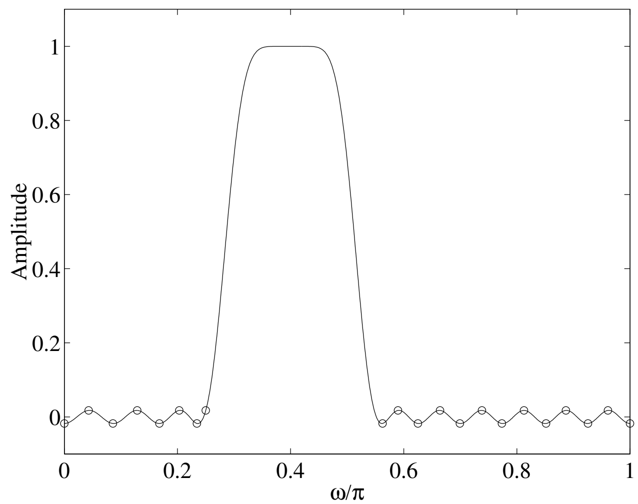

Figure 4: $N = 55$, $L = 16$, $\omega_p = 0.4\pi$, $\omega_t = 0.15\pi$

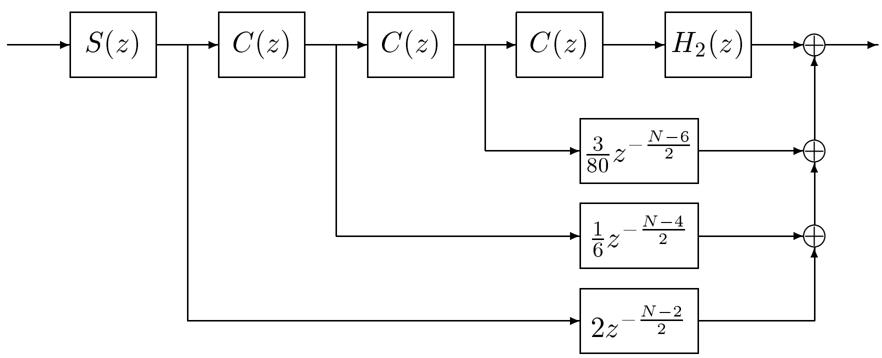

Figure 5: Filter structure for implementation and design of a length N lowpass differentiator. $C(z) = -(1 - z^{-1})^2/2$, $S(z) = (z^{-1} - 1)/2$, N even.

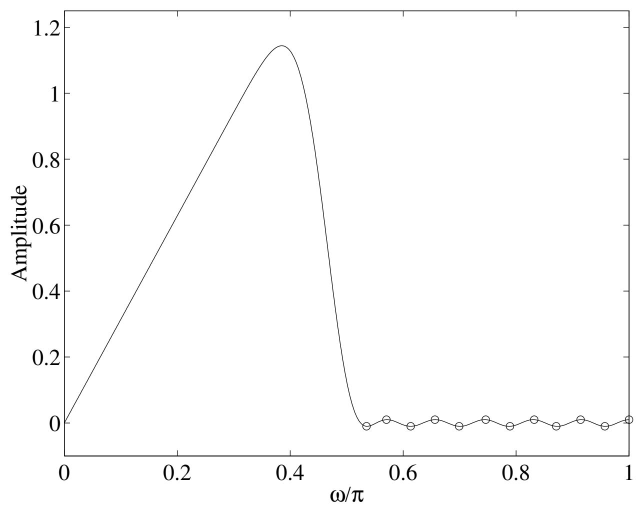

Figure 6: $N = 58$, $L = 34$, $\delta_s = 0.01$