Data-driven model reduction:

The Loewner framework for linear and nonlinear systems

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Outline

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The Loewner matrix

Given: a row array $(\mu_j, \mathbf{v}_j)$, $j = 1, \cdots, k$, and
a column array $(\lambda_i, \mathbf{w}_i)$, $i = 1, \cdots, q$,

the associated **Loewner** or **divided-differences matrix** is:

$$\mathbb{L}(\{\mu_i\}, \{\lambda_i\}) = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_q}{\mu_1 - \lambda_q} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_k - \mathbf{w}_1}{\mu_k - \lambda_1} & \cdots & \frac{\mathbf{v}_k - \mathbf{w}_q}{\mu_k - \lambda_q} \end{bmatrix} \in \mathbb{C}^{k \times q}$$

- If there is a known underlying function $\mathbf{g}$, then $\mathbf{w}_i = \mathbf{g}(\lambda_i)$, and $\mathbf{v}_j = \mathbf{g}(\mu_j)$.
- If $\mu_i = \lambda_i \Rightarrow (\mathbb{L})_{i,i} = \mathbf{w}_i^{(1)}$, which in case of a known underlying function $\mathbf{g}$ becomes: $\mathbf{w}_i^{(1)} = \frac{d\mathbf{g}}{ds}|_{s=\lambda_i}$.

K. Loewner: Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

Outline

- Lagrange basis for space of polynomials of degree at most n: given $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n+1$: $\lambda_i \neq \lambda_j$, $i \neq j$,

$$\mathbf{q}_i(s) := \Pi_{i' \neq i}(s - \lambda_{i'}),\ i = 1, \cdots, n+1,$$

- For given constants α_i, $\mathbf{w}_i$, $i = 1, \cdots, n+1$, consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \quad \alpha_i \neq 0.$$

- Solving for $\mathbf{g}$ we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}}$$

It follows that: $\mathbf{g}(\lambda_i) = \mathbf{w}_i$

This is the **barycentric Lagrange interpolation** formula.

The free parameters α_i, can be determined so that additional constraints are satisfied:

$$\mathbf{g}(\mu_j) = \mathbf{v}_j, \; j = 1, \cdots, r,$$

where $(\mu_j, \mathbf{v}_j)$, $\mu_i \neq \mu_j$, are given. For this to hold $\mathbb{L}\mathbf{c} = 0$, where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \cdots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r \times (n+1)}, \; \mathbf{c} = \begin{bmatrix} \alpha_1, \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

$\mathbb{L}$: **Loewner matrix** with row array $(\mu_j, \mathbf{v}_j), \; j = 1, \cdots, r,$ and column array $(\lambda_i, \mathbf{w}_i), \; i = 1, \cdots, n+1$.

Main property

Let $\mathbb{L}$ be a $p \times k$ Loewner matrix. Then

$$p, k \geq \deg \mathbf{g} \quad \Rightarrow \quad \operatorname{rank} \mathbb{L} = \deg \mathbf{g}.$$

State-space representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \ \ \mathbf{r}_2, \ \ \cdots \ \ \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \ \ \mathbf{w}_2 \ \ \cdots \ \ \mathbf{w}_k] \in \mathbb{C}^{p \times k}, \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

State-space representation: the Loewner matrix pair

Recall data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$, $\ell_j \mathbf{H}(\mu_j) = \mathbf{v}_j$.

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q \times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}}$$

If $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, is given:
$\mathbf{x}_i := (\lambda_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{r}_i$, $\mathbf{y}_j := \ell_j\mathbf{C}(\mu_j\mathbf{E} - \mathbf{A})^{-1} \Rightarrow$
$\mathbf{X}$, $\mathbf{Y}$: *gen. reach./observ. matrix* $\Rightarrow$

$$\Rightarrow \boxed{\mathbb{L} = -\mathbf{Y}\mathbf{E}\mathbf{X}}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q \times k}$ is:

$$\mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1\lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k\lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1\lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k\lambda_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}_\sigma$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}_\sigma\Lambda - M\mathbb{L}_\sigma = M\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\Lambda}$$

$\mathbb{L}_\sigma$ can be factored as

$$\Rightarrow \boxed{\mathbb{L}_\sigma = -\mathbf{Y}\mathbf{A}\mathbf{X}}$$

Construction of Interpolants (Models)

- If the pencil $(\mathbb{L}_\sigma, \mathbb{L})$ is regular, then

$$\mathbf{E} = -\mathbb{L}, \;\; \mathbf{A} = -\mathbb{L}_\sigma, \;\; \mathbf{B} = \mathbf{V}, \;\; \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e., $\mathbf{H}(s)$ interpolates the data:

$$\boxed{\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_\sigma - s\mathbb{L})^{-1}\mathbf{V}}$$

- Otherwise, if the **numerical** $\operatorname{rank} \mathbb{L} = k$, compute the rank revealing *SVD*:

$$\boxed{\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*}$$

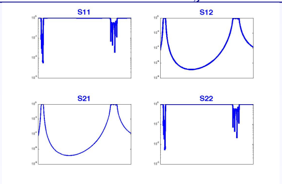
Theorem. A realization [**E**, **A**, **B**, **C**], of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_\sigma\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

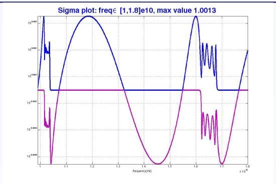
Remark. If we have more data than necessary, we can consider $(\mathbb{L}_\sigma, \; \mathbb{L}, \; \mathbf{V}, \; \mathbf{W})$, as a singular model of the data. **Consequence**: The original pencil $(\mathbb{L}_\sigma, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

An example: 1001 S-parameter measurements between 10-18 GHz provided (CST)

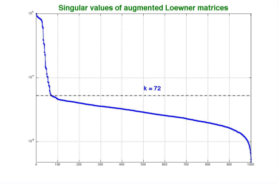
Data frequency response $\|\mathbf{S}_{i,j}\|$, $i, j = 1, 2$.



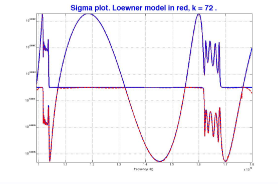
Data two singular values.



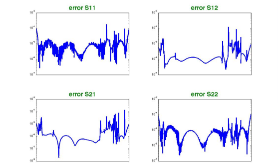
Singular values of 1001 $\times$ 1001 Loewner matrix



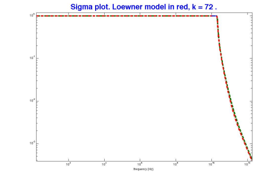
Singular-value fit of model $k = 72$



S-parameter-error: $\in [10^{-6}, 10^{-4}]$



Two singular values of model: $\omega \in [0, 10\text{THz}]$



Outline

Generalization: the Loewner matrix for parameter-dependent systems

- Consider $\mathbf{g}(s,t)$ defined by

$$\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\alpha_{i,j}\,\frac{\mathbf{g}-\mathbf{w}_{i,j}}{(s-\lambda_i)(t-\pi_j)}=0,\ \alpha_{i,j}\neq 0 \quad\Rightarrow\quad \mathbf{g}(s,t)=\frac{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\frac{\alpha_{i,j}\mathbf{w}_{i,j}}{(s-\lambda_i)(t-\pi_j)}}{\sum_{i=1}^{n+1}\sum_{j=1}^{m+1}\frac{\alpha_{i,j}}{(s-\lambda_i)(t-\pi_j)}}.$$

This is the two-variable barycentric interpolation formula. Therefore $\mathbf{g}$ satisfies the interpolation conditions $\mathbf{g}(\lambda_i,\pi_j)=\mathbf{w}_{i,j}$.

- The generalized Loewner matrix has the form:

$$\ell_{i,j}^{k,l}:=\frac{\mathbf{v}_{k,l}-\mathbf{w}_{i,j}}{(\mu_k-\lambda_i)(\nu_l-\pi_j)}.$$

Given $\mathbf{g}(s,t)$ of complexity (n,m), construct $\mathbb{L}$, by means of its samples. Assume also that $n'\geq n+1$ and $m'\geq m+1$. The rank of the associated Loewner matrix $\mathbb{L}$ is:

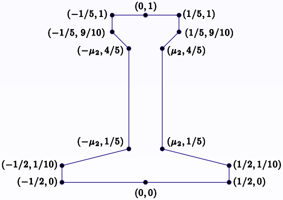
$$\mathrm{rank}\,\mathbb{L}\ =\ \mathbf{n}'\mathbf{m}'-(\mathbf{n}'-\mathbf{n})(\mathbf{m}'-\mathbf{m})$$

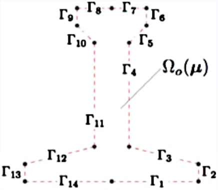
$$\boxed{\textbf{Property generalized: barycentric formula}}$$

Thermal treatment of railroad rail (Quarteroni, Rozza, Manzoni)

$$
\begin{aligned}
\frac{\partial \mathbf{u}(\mu,t)}{\partial t} - \Delta \mathbf{u}(\mu,t) &= 0 && \text{in } \Omega_0(\mu) \\
\mathbf{u}(\mu, t=0) &= 0 && \text{in } \Omega_0(\mu) \\
\frac{\partial \mathbf{u}(\mu,t)}{\partial \mathbf{n}} + \mu_1 \mathbf{u}(\mu,t) &= \mu_1 \mathbf{g}(t) && \text{on } \partial\Omega_0(\mu)
\end{aligned}
$$

Boundary conditions: heat exchange between fluid of thermal treatment and rail. Spatial discretization results in $N = 16737$ degrees of freedom.





$$\mathbf{A} = \begin{cases} \frac{-400\mu_2^2+80\mu_2-13}{30\mu_2-12}\,\mathbf{A}_1 - \frac{4}{10\mu_2-5}\,\mathbf{A}_2 + \frac{1}{10\mu_2}\,\mathbf{A}_3 + \\ \frac{\mu_1\sqrt{5}}{15}\sqrt{\frac{\left|10000\mu_2^4-16000\mu_2^3+8400\mu_2^2-1600\mu_2+109\right|}{20\mu_2^2-8\mu_2+1}}\,\mathbf{A}_4 + \\ \frac{\mu_1\sqrt{2}}{8}\sqrt{\frac{641-5000\mu_2+15000\mu_2^2-20000\mu_2^3+10000\mu_2^4}{50\mu_2^2-50\mu_2+13}}\,\mathbf{A}_5 + \mathbf{A}_6 + \mu_1\mathbf{A}_7 + \mu_2\mathbf{A}_8 + \mu_2^2\mathbf{A}_9 \end{cases}$$

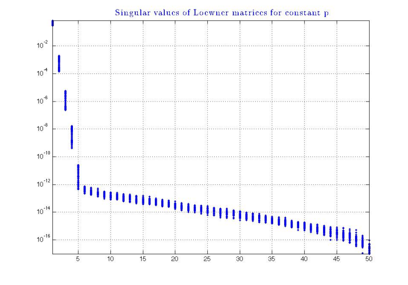
$$\mathbf{B} = \begin{cases} \frac{\mu_1\sqrt{5}}{15}\sqrt{\frac{\left|10000\mu_2^4-16000\mu_2^3+8400\mu_2^2-1600\mu_2+109\right|}{20\mu_2^2-8\mu_2+1}}\,\mathbf{B}_1 + \\ \frac{\mu_1\sqrt{2}}{8}\sqrt{\frac{641-5000\mu_2+15000\mu_2^2-20000\mu_2^3+10000\mu_2^4}{50\mu_2^2-50\mu_2+13}}\,\mathbf{B}_2 + \mu_1\,\mathbf{B}_3 \end{cases}$$

$$\mathbf{C} = \frac{1}{N}\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}, \quad \mathbf{E} = \mathbf{E}_1 + \mu_2\mathbf{E}_2$$

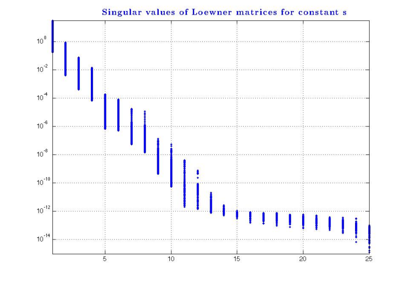
$$\mathbf{\Phi}(s;\mu_1,\mu_2) = \mathbf{C}\left[s\,\mathbf{E}(\mu_1,\mu_2) - \mathbf{A}(\mu_1,\mu_2)\right]^{-1}\mathbf{B}(\mu_1,\mu_2)$$

We take 50 measurements in s and 50 in μ_2.

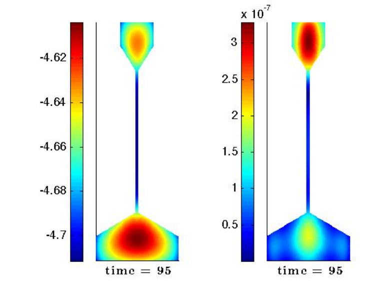
Singular values of $\mathbb{L}$ for constant μ_2



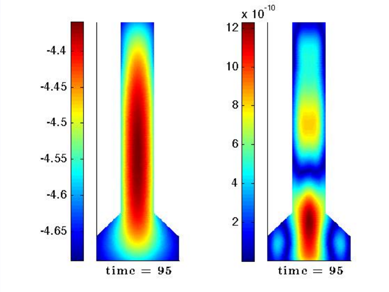
Singular values of $\mathbb{L}$ for constant s



Parameter values = $(10, 0.02)$



Parameter values = $(10, 0.20)$



Using the 2D Loewner matrix framework, reduced systems with complexity 5 in the frequency and 10 in the parameter μ_2 (left and right panes) were computed.

Outline

We will now consider (scalar) dynamical systems described by

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \ \mathbf{y} \in \mathbb{R}$. Such systems are equivalent to an inifnite sequence of rational functions, namely:

$$\begin{aligned} \mathbf{H}_1(s_1) &= \mathbf{C}\left[s_1\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{B} \\ \mathbf{H}_2(s_1, s_2) &= \mathbf{C}\left[s_1\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{N}\left[s_2\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{B} \\ \mathbf{H}_3(s_1, s_2, s_3) &= \mathbf{C}\left[s_1\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{N}\left[s_2\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{N}\left[s_3\mathbf{E} - \mathbf{A}\right]^{-1}\mathbf{B} \\ &\vdots \end{aligned}$$

These rational functions allow us to treat model reduction of such systems as (approximate) rational interpolation.

Recall the Loewner matrix pair for (scalar) linear systems:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_k - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}, \ \mathbb{L}_\sigma = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1 - \lambda_1 \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 \mathbf{v}_k - \lambda_k \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q - \lambda_1 \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q \mathbf{v}_q - \lambda_k \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

Suppose that the underlying transfer function is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

Let $\mathcal{O}$, $\mathcal{R}$ be such that

$$\mathcal{O}(j,:) = \mathbf{C}(\mu_j \mathbf{E} - \mathbf{A})^{-1}, \quad \mathcal{R}(:,i) = (\lambda_i \mathbf{E} - \mathbf{A})^{-1}\mathbf{B},$$

then, $\mathcal{R}$ is the generalized reachability matrix and $\mathcal{O}$ is the generalized observability matrix associated with the given data. It follows that

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L},$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E}-\mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q\mathbf{E}-\mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\left[\ (\lambda_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{B} \quad \cdots \quad (\lambda_k\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\ \right]}_{\mathcal{R}} = -\mathbb{L}_\sigma,$$

In addition $\mathbf{V} = \mathbf{C}\mathbf{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$. Consequently the fundamental relationships to be generalized are:

$$\boxed{\mathbb{L} = -\mathcal{O}\mathbf{E}\mathcal{R},\ \ \mathbb{L}_\sigma = -\mathcal{O}\mathbf{A}\mathcal{R},\ \ \mathbf{V} = \mathbf{C}\mathcal{R},\ \ \mathbf{W} = \mathcal{O}\mathbf{B}}$$

where

$$\boxed{\mathcal{O} = \begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E}-\mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q\mathbf{E}-\mathbf{A})^{-1} \end{bmatrix},\ \ \mathcal{R} = \left[\ (\lambda_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{B} \quad \cdots \quad (\lambda_k\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\ \right]}$$

and

Left array $\{(\mu_j, \mathbf{v}_j),\ j = 1, \cdots, q\}$; Right array $\{(\lambda_i, \mathbf{w}_i),\ i = 1, \cdots, k\}$.

Fact: if $q = k = N \Rightarrow 2N$ moments of $\mathbf{H}$ are matched

$$\boxed{\textbf{Property generalized: factorization of } \mathbb{L},\ \mathbb{L}_\sigma,\ \mathbf{V},\ \mathbf{W}}$$

Simple example: consider ordered tuples of left and right interpolation points:

$$\begin{bmatrix} (\mu_1) \\ (\mu_1,\mu_2) \end{bmatrix}, \quad \begin{bmatrix} (\lambda_1), & (\lambda_2,\lambda_1) \end{bmatrix} \Rightarrow$$

the generalized observability/controllability matrices are:

$$\mathcal{O}=\begin{bmatrix} \mathbf{C}(\mu_1\mathbf{E}-\mathbf{A})^{-1} \\ \mathbf{C}(\mu_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}(\mu_2\mathbf{E}-\mathbf{A})^{-1} \end{bmatrix}, \mathcal{R}=\left[(\lambda_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}, \quad (\lambda_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}(\lambda_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{B}\right].$$

Then

$$\mathcal{O}\mathbf{E}\mathcal{R}=\begin{bmatrix} \frac{\mathbf{H}_1(\mu_1)-\mathbf{H}_1(\lambda_1)}{\mu_1-\lambda_1} & \frac{\mathbf{H}_2(\mu_1,\lambda_1)-\mathbf{H}_2(\lambda_2,\lambda_1)}{\mu_1-\lambda_2} \\ \frac{\mathbf{H}_2(\mu_1,\mu_2)-\mathbf{H}_2(\mu_1,\lambda_1)}{\mu_2-\lambda_1} & \frac{\mathbf{H}_3(\mu_1,\mu_2,\lambda_1)-\mathbf{H}_3(\mu_1,\lambda_2,\lambda_1)}{\mu_2-\lambda_2} \end{bmatrix}=\hat{\mathbf{E}},$$

$$\mathcal{O}\mathbf{A}\mathcal{R}=\begin{bmatrix} \frac{\mu_1\mathbf{H}_1(\mu_1)-\lambda_1\mathbf{H}_1(\lambda_1)}{\mu_1-\lambda_1} & \frac{\mu_1\mathbf{H}_2(\mu_1,\lambda_1)-\lambda_2\mathbf{H}_2(\lambda_2,\lambda_1)}{\mu_1-\lambda_2} \\ \frac{\mu_2\mathbf{H}_2(\mu_1,\mu_2)-\lambda_1\mathbf{H}_2(\mu_1,\lambda_1)}{\mu_2-\lambda_1} & \frac{\mu_2\mathbf{H}_3(\mu_1,\mu_2,\lambda_1)-\lambda_2\mathbf{H}_3(\mu_1,\lambda_2,\lambda_1)}{\mu_2-\lambda_2} \end{bmatrix}=\hat{\mathbf{A}},$$

$$\mathcal{O}\mathbf{N}\mathcal{R}=\begin{bmatrix} \mathbf{H}_2(\mu_1,\lambda_1) & \mathbf{H}_3(\mu_1,\lambda_2,\lambda_{11}) \\ \mathbf{H}_3(\mu_1,\mu_2,\lambda_1) & \mathbf{H}_4(\mu_1,\mu_2,\lambda_2,\lambda_1) \end{bmatrix}=\hat{\mathbf{N}},$$

$$\mathbf{C}\mathcal{R}=\begin{bmatrix} \mathbf{H}_1(\lambda_1), & \mathbf{H}_2(\lambda_2,\lambda_1) \end{bmatrix}=\hat{\mathbf{C}}, \quad \mathcal{O}\mathbf{B}=\begin{bmatrix} \mathbf{H}_1(\mu_1) \\ \mathbf{H}_2(\mu_1,\mu_2) \end{bmatrix}=\hat{\mathbf{B}}.$$

Moments matched $\mathbf{H}_1(\mu_1), \mathbf{H}_1(\lambda_1), \mathbf{H}_2(\mu_1,\mu_2), \mathbf{H}_2(\mu_1,\lambda_1), \mathbf{H}_2(\lambda_2,\lambda_1),$
$\mathbf{H}_3(\mu_1,\mu_2,\lambda_1), \mathbf{H}_3(\mu_1,\lambda_2,\lambda_1), \mathbf{H}_4(\mu_1,\mu_2,\lambda_2,\lambda_1).$

Theorem

Recall that a bilinear system is completely described by means of the rational functions:

$$\mathbf{H}_m(\nu_1, \nu_2, \cdots, \nu_m) = \mathbf{C}\,(\nu_1\mathbf{E}-\mathbf{A})^{-1}\,\mathbf{N}\,(\nu_2\mathbf{E}-\mathbf{A})^{-1}\,\mathbf{N}\;\cdots\;\mathbf{N}\,(\nu_m\mathbf{E}-\mathbf{A})^{-1}\,\mathbf{B},\;\; m = 1, 2, \cdots .$$

(a) Selecting the data.

Given the k^{th} ordered row and the ℓ^{th} ordered column tuples

$$(\mu_1, \cdots, \mu_{k-1}, \mu_k) \quad \text{and} \quad (\lambda_\ell, \lambda_{\ell-1}, \cdots, \lambda_1),$$

we need the following data:

$$\mathbf{H}_k(\mu_1, \cdots, \mu_{k-1}, \mu_k),\; \mathbf{H}_\ell(\lambda_\ell, \lambda_{\ell-1}, \cdots, \lambda_1),\; \mathbf{H}_{k+\ell}(\mu_1, \cdots, \mu_{k-1}, \mu_k, \lambda_\ell, \lambda_{\ell-1}, \cdots, \lambda_1) \text{ and}$$

$$\mathbf{H}_{k+\ell-1}(\mu_1, \cdots, \mu_{k-1}, \lambda_\ell, \lambda_{\ell-1}, \cdots, \lambda_1), \quad \mathbf{H}_{k+\ell-1}(\mu_1, \cdots, \mu_{k-1}, \mu_k, \lambda_{\ell-1}, \cdots, \lambda_1)$$

(b) The generalized Loewner matrices.

The first element above is $\mathbf{V}(k, 1)$, the second is $\mathbf{W}(1, \ell)$, the third is $\boldsymbol{\Psi}(k, \ell)$, while

$$\mathbb{L}(j, i) = \frac{\mathbf{H}_{j+i-1}(\mu_1, \cdots, \mu_j, \lambda_{i-1}, \cdots, \lambda_1) - \mathbf{H}_{j+i-1}(\mu_1, \cdots, \mu_{j-1}, \lambda_i, \cdots, \lambda_1)}{\mu_j - \lambda_i},$$

$$\mathbb{L}_\sigma(j, i) = \frac{\mu_j\mathbf{H}_{j+i-1}(\mu_1, \cdots, \mu_j, \lambda_{i-1}, \cdots, \lambda_1) - \lambda_i\mathbf{H}_{j+i-1}(\mu_1, \cdots, \mu_{j-1}, \lambda_i, \cdots, \lambda_1)}{\mu_j - \lambda_i}$$

(c) The reduced system of order N matches $N^2 + 2N$ moments of the original system. ■

Review: Optimal $\mathcal{H}_2$ model reduction for linear systems

Recall: the $\mathcal{H}_2$ norm of a stable system $\mathbf{\Sigma}$ is:

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{trace}\ [\mathbf{H}(i\omega)\mathbf{H}^*(-i\omega)]\ d\omega\right)^{1/2},$$

where $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, is the system transfer function.

Goal: construct a *Krylov projector* such that $\mathbf{\Sigma}_k = \underset{\deg(\hat{\mathbf{\Sigma}})=k}{\arg\min} \left\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\right\|_{\mathcal{H}_2}$.

The optimization problem is **nonconvex**. We seek reduced models that satisfy **first-order necessary optimality conditions**:

Let $\mathbf{H}_k$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote its poles. Assuming for simplicity that $m = p = 1$, the following **interpolation conditions** hold. Thus the optimal reduced system $\mathbf{H}_k$ matches the first two moments of the original system at the **mirror image of its poles**:

$$\mathbf{H}(-\hat{\lambda}_i^*) = \mathbf{H}_k(-\hat{\lambda}_i^*) \quad \text{and} \quad \frac{d}{ds}\mathbf{H}(s)\Big|_{s=-\hat{\lambda}_i^*} = \frac{d}{ds}\mathbf{H}_k(s)\Big|_{s=-\hat{\lambda}_i^*}$$

Numerical algorithm

1. Make an initial selection of σ_i, for $i = 1, \cdots, k$
2. $\mathbf{W} = [(\sigma_1\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
3. $\mathbf{V} = [(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}]$
4. while (not converged)
   - $\mathbf{E}_k = \mathbf{W}^*\mathbf{E}\mathbf{V}$, $\mathbf{A}_k = \mathbf{W}^*\mathbf{A}\mathbf{V}$,
   - $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_k, \mathbf{E}_k)$ + Newton correction, $i = 1, \cdots, k$
   - $\mathbf{W} = [(\sigma_1\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
   - $\mathbf{V} = [(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}]$
5. $\mathbf{E}_k = \mathbf{W}^*\mathbf{E}\mathbf{V}$, $\mathbf{A}_k = \mathbf{W}^*\mathbf{A}\mathbf{V}$, $\mathbf{B}_k = \mathbf{W}^*\mathbf{B}$, $\mathbf{C}_k = \mathbf{C}\mathbf{V}$

Rational Krylov projections and Sylvester equations for linear systems

Given the projection matrix

$$\mathbf{V} = [(\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}] \in \mathbb{C}^{n\times k},$$

and $\tilde{\mathbf{V}} \in \mathbb{C}^{n\times k}$ satisfying the Sylvester equation

$$\mathbf{A}\tilde{\mathbf{V}} + \mathbf{E}\tilde{\mathbf{V}}\mathbf{\Lambda} + \mathbf{B}\hat{\mathbf{B}}^* = \mathbf{0}, \quad \mathbf{\Lambda} = \mathrm{diag}\,[\lambda_1, \cdots, \lambda_k], \quad \hat{\mathbf{B}} \in \mathbb{R}^k,$$

are related by

$$\text{span col}\,\mathbf{V} = \text{span col}\,\tilde{\mathbf{V}}$$

In other words, both $\mathbf{V}$ and $\tilde{\mathbf{V}}$ achieve the same result (interpolation of the transfer function of the reduced system at λ_i, $i = 1, \cdots, k$).

Question: can the $\mathcal{H}_2$ reduction method be extended to bilinear system?

Answer: Yes, see: Benner-Breiten, Flagg-Gugercin.

Tool: Sylvester equation.

Consider the (scalar) bilinear system

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \;\; \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}, \; \mathbf{B}, \mathbf{C}^* \in \mathbb{R}^{n}.$$

The following generalized **Lyapunov** equation is associated:

$$\mathbf{A}\mathcal{P}\mathbf{E}^* + \mathbf{E}\mathcal{P}\mathbf{A}^* + \mathbf{N}\mathcal{P}\mathbf{N}^* + \mathbf{B}\mathbf{B}^* = \mathbf{0}.$$

To reduce this system we will look at projections $\mathbf{V} \in \mathbb{C}^{n\times k}$, which satisfy the **Sylvester** equation:

$$\mathbf{A}\mathbf{V} + \mathbf{E}\mathbf{V}\mathbf{\Lambda} + \mathbf{N}\mathbf{V}\mathbf{U} + \mathbf{B}\mathbf{z}^* = \mathbf{0}, \quad \mathbf{\Lambda} = \mathrm{diag}[\lambda_1, \cdots, \lambda_k], \;\; \mathbf{U} \in \mathbb{C}^{k\times k}, \;\; \mathbf{z} \in \mathbb{C}^{k}.$$

To get the general idea of what is interpolated, we discuss the **simple special case** $k = 1$.

Rename $\mathbf{\Lambda} = -\lambda$, $\mathbf{U} = \alpha$, and assume without loss of generality that $\mathbf{z} = 1$:

$$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\lambda + \mathbf{N}\mathbf{V}\alpha + \mathbf{B} = \mathbf{0} \;\; \Rightarrow \;\; \boxed{\mathbf{V} = (\lambda\mathbf{E} - \mathbf{A} - \mathbf{N}\alpha)^{-1}\,\mathbf{B}}$$

Similarly, let $\mathbf{W} \in \mathbb{C}^{n\times k}$ satisfy

$$\mathbf{A}^*\mathbf{W} - \mathbf{E}^*\mathbf{W}\mu^* + \beta^*\mathbf{N}^*\mathbf{W} + \mathbf{C}^* = \mathbf{0} \;\; \Rightarrow \;\; \boxed{\mathbf{W} = \left(\mu^*\mathbf{E}^* - \mathbf{A}^* - \beta^*\mathbf{N}^*\right)^{-1}\mathbf{C}^*}$$

Define the transfer functions:

$$\begin{aligned} \mathbf{G}_1(s_1, t_1) &= \mathbf{C}(s_1\mathbf{E} - \mathbf{A} - t_1\mathbf{N})^{-1}\mathbf{B}, \\ \mathbf{G}_2(s_1, s_2, t_1, t_2) &= \mathbf{C}(s_1\mathbf{E} - \mathbf{A} - t_1\mathbf{N})^{-1}\mathbf{N}(s_2\mathbf{E} - \mathbf{A} - t_2\mathbf{N})^{-1}\mathbf{B} \end{aligned}$$

The reduced system is:

$$\hat{\mathbf{C}} = \mathbf{CV}, \quad \hat{\mathbf{E}} = \mathbf{W}^*\mathbf{EV}, \quad \hat{\mathbf{A}} = \mathbf{W}^*\mathbf{AV}, \quad \hat{\mathbf{N}} = \mathbf{W}^*\mathbf{NV}, \quad \hat{\mathbf{B}} = \mathbf{W}^*\mathbf{B}.$$

Then, the following **interpolation conditions** are satisfied:

- $\mathbf{G}_1(\lambda, \alpha) = \hat{\mathbf{G}}_1(\lambda, \alpha)$
- $\mathbf{G}_1(\mu, \beta) = \hat{\mathbf{G}}_1(\mu, \beta)$
- $\mathbf{G}_2(\mu, \lambda, \beta, \alpha) = \hat{\mathbf{G}}_2(\mu, \lambda, \beta, \alpha)$

With $\mathbf{\Phi}(s) = s\mathbf{E} - \mathbf{A}$, we have

$$(\lambda\mathbf{E} - \mathbf{A} - \mathbf{N}\alpha)^{-1} = (\mathbf{\Phi}(\lambda) - \mathbf{N}\alpha)^{-1} = (\mathbf{I} - \overbrace{\alpha\mathbf{\Phi}(\lambda)^{-1}\mathbf{N}}^{\mathbf{K}})^{-1}\mathbf{\Phi}(\lambda)^{-1} = (\mathbf{I} - \mathbf{K})^{-1}\mathbf{\Phi}^{-1} = \left[\mathbf{I} + \mathbf{K} + \mathbf{K}^2 + \cdots\right]\mathbf{\Phi}^{-1},$$

provided that $\mathbf{K}$ is contractive. Hence:

The interpolation conditions associated with the Sylvester equations, can be expressed in terms of the original transfer functions $\mathbf{H}_k(s_1, \cdots, s_k)$, by means of the associated weighted Volterra series $\Rightarrow$ Volterra Series Interpolation.

- $\mathbf{G}_1(\lambda, \alpha) = \sum_{k\geq 0} \alpha^k \mathbf{H}_{k+1}(\lambda, \cdots, \lambda) = \sum_{k\geq 0} \alpha^k \hat{\mathbf{H}}_{k+1}(\lambda, \cdots, \lambda) = \hat{\mathbf{G}}_1(\lambda, \alpha)$
- $\mathbf{G}_1(\mu, \beta) = \sum_{k\geq 0} \beta^k \mathbf{H}_{k+1}(\mu, \cdots, \mu) = \sum_{k\geq 0} \beta^k \hat{\mathbf{H}}_{k+1}(\mu, \cdots, \mu) = \hat{\mathbf{G}}_1(\mu, \beta)$,
- $\mathbf{G}_2(\mu, \lambda, \beta, \alpha) = \left[\sum_{k\geq 0} \alpha^k \mathbf{H}_{k+1}(\lambda, \cdots, \lambda)\right] \mathbf{N} \left[\sum_{k\geq 0} \alpha^k \mathbf{H}_{k+1}(\lambda, \cdots, \lambda)\right] =$

$$= \left[\sum_{k\geq 0} \alpha^k \mathbf{H}_{k+1}(\lambda, \cdots, \lambda)\right] \mathbf{N} \left[\sum_{k\geq 0} \alpha^k \mathbf{H}_{k+1}(\lambda, \cdots, \lambda)\right] = \hat{\mathbf{G}}_2(\mu, \lambda, \beta, \alpha)$$

Conculsion: **optimal $\mathcal{H}_2$ for bilinear systems is performed by iterative solution of the associated bilinear Sylvester equations.**

Quadratic nonlinear systems

Consider

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Q}\,\mathbf{x}(t) \otimes \mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t).$$

We introduce the following set of multivariate rational functions, where $\boldsymbol{\Phi}_s = (s\,\mathbf{E} - \mathbf{A})^{-1}$:

$$\begin{aligned}
\mathbf{H}_1(s_1) &= \mathbf{C}\boldsymbol{\Phi}_{s_1}\mathbf{B} \\
\mathbf{H}_2(s_1, s_2, s_3) &= \mathbf{C}\boldsymbol{\Phi}_{s_3}\mathbf{Q}(\boldsymbol{\Phi}_{s_1}\mathbf{B} \otimes \boldsymbol{\Phi}_{s_2}\mathbf{B}) \\
\mathbf{H}_3(s_1, s_2, s_3, s_4, s_5) &= \mathbf{C}\boldsymbol{\Phi}_{s_1}\mathbf{Q}\{[\boldsymbol{\Phi}_{s_2}\mathbf{Q}(\boldsymbol{\Phi}_{s_3}\mathbf{B} \otimes \boldsymbol{\Phi}_{s_4}\mathbf{B})] \otimes \boldsymbol{\Phi}_{s_5}\mathbf{B}\} \\
&= \mathbf{C}\boldsymbol{\Phi}_{s_1}\mathbf{Q}(\boldsymbol{\Phi}_{s_5}\mathbf{B} \otimes \boldsymbol{\Phi}_{s_2})\mathbf{Q}(\boldsymbol{\Phi}_{s_3}\mathbf{B} \otimes \boldsymbol{\Phi}_{s_4}\mathbf{B})
\end{aligned}$$

Example. Given the ordered tuples of left/right interpolation points

$$\begin{bmatrix} (\mu_1) \\ (\mu_1, \mu_2, \mu_3) \end{bmatrix}, \quad [(\lambda_1), \ (\lambda_2)]$$

The goal is to recover $(\mathbf{A}, \mathbf{Q}, \mathbf{B}, \mathbf{C})$ from measurements of the above rational functions.

We define appropriate generalized observability and controllability matrices:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}\boldsymbol{\Phi}_{\mu_1} \\ \mathbf{C}\boldsymbol{\Phi}_{\mu_1}\mathbf{Q}\,(\boldsymbol{\Phi}_{\mu_2}\mathbf{B} \otimes \boldsymbol{\Phi}_{\mu_3}) \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} \boldsymbol{\Phi}_{\lambda_1}\mathbf{B}, & \boldsymbol{\Phi}_{\lambda_2}\mathbf{B} \end{bmatrix}.$$

It follows that

$$\mathcal{O}\mathbf{E}\mathcal{R} = \begin{bmatrix} \frac{\mathbf{H}_1(\mu_1)-\mathbf{H}_1(\lambda_1)}{\mu_1-\lambda_1} & \frac{\mathbf{H}_1(\mu_1)-\mathbf{H}_1(\lambda_2)}{\mu_1-\lambda_2} \\ \frac{\mathbf{H}_2(\mu_1,\mu_2,\mu_3)-\mathbf{H}_2(\mu_1,\mu_2,\lambda_1)}{\mu_3-\lambda_1} & \frac{\mathbf{H}_2(\mu_1,\mu_2,\mu_3)-\mathbf{H}_2(\mu_1,\mu_2,\lambda_2)}{\mu_3-\lambda_2} \end{bmatrix} = \hat{\mathbf{E}},$$

$$\mathcal{O}\mathbf{A}\mathcal{R} = \begin{bmatrix} \frac{\mu_1\mathbf{H}_1(\mu_1)-\lambda_1\mathbf{H}_1(\lambda_1)}{\mu_1-\lambda_1} & \frac{\mu_1\mathbf{H}_1(\mu_1)-\lambda_2\mathbf{H}_1(\lambda_2)}{\mu_1-\lambda_2} \\ \frac{\mu_3\mathbf{H}_2(\mu_1,\mu_2,\mu_3)-\lambda_1\mathbf{H}_2(\mu_1,\mu_2,\lambda_1)}{\mu_3-\lambda_1} & \frac{\mu_3\mathbf{H}_2(\mu_1,\mu_2,\mu_3)-\lambda_2\mathbf{H}_2(\mu_1,\mu_2,\lambda_2)}{\mu_3-\lambda_2} \end{bmatrix} = \hat{\mathbf{A}}$$

$$\mathcal{O}\mathbf{B} = \begin{bmatrix} \mathbf{H}_1(\mu_1) \\ \mathbf{H}_2(\mu_1,\mu_2,\mu_3) \end{bmatrix} = \hat{\mathbf{B}}, \quad \mathbf{C}\mathcal{R} = \begin{bmatrix} \mathbf{H}_1(\lambda_1) & \mathbf{H}_1(\lambda_2) \end{bmatrix} = \hat{\mathbf{C}} \qquad \text{and}$$

$$\mathcal{O}\,\mathbf{Q}\,(\mathcal{R}\otimes\mathcal{R}) =$$

$$= \begin{bmatrix} \mathbf{H}_2(\mu_1,\lambda_1,\lambda_1) & \mathbf{H}_2(\mu_1,\lambda_1,\lambda_2) & \mathbf{H}_2(\mu_1,\lambda_2,\lambda_1) & \mathbf{H}_2(\mu_1,\lambda_2,\lambda_2) \\ \mathbf{H}_3(\mu_1,\mu_3,\lambda_1,\lambda_1,\mu_2) & \mathbf{H}_3(\mu_1,\mu_3,\lambda_1,\lambda_2,\mu_2) & \mathbf{H}_3(\mu_1,\mu_3,\lambda_2,\lambda_1,\mu_2) & \mathbf{H}_3(\mu_1,\mu_3,\lambda_2,\lambda_2,\mu_2) \end{bmatrix} = \hat{\mathbf{Q}}$$

The following $2n+n^3=12$ moments are matched

3 moments of $\mathbf{H}_1$: $\mathbf{H}_1(\mu_1)$, $\mathbf{H}_1(\lambda_1)$, $\mathbf{H}_1(\lambda_2)$

5 moments of $\mathbf{H}_2$: $\mathbf{H}_2(\mu_1,\mu_2,\mu_3)$, $\mathbf{H}_2(\mu_1,\lambda_1,\lambda_1)$, $\mathbf{H}_2(\mu_1,\lambda_2,\lambda_1)$, $\mathbf{H}_2(\mu_1,\lambda_1,\lambda_2)$, $\mathbf{H}_2(\mu_1,\lambda_2,\lambda_2)$.

4 moments of $\mathbf{H}_3$: $\mathbf{H}_3(\mu_1,\mu_3,\lambda_1,\lambda_1,\mu_2)$, $\mathbf{H}_3(\mu_1,\mu_3,\lambda_1,\lambda_2,\mu_2)$, $\mathbf{H}_3(\mu_1,\mu_3,\lambda_2,\lambda_1,\mu_2)$, $\mathbf{H}_3(\mu_1,\mu_3,\lambda_2,\lambda_2,\mu_2)$.

Burgers' equation (Breiten, Damm)

$$\frac{\partial w(x,t)}{\partial t} + w(x,t)\frac{\partial w(x,t)}{\partial x} = \nu\,\frac{\partial^2 w(x,t)}{\partial x^2}, \quad (x,t) \in [0,L]\times[0,T],$$

where ν is the viscosity assumed constant, and the following conditions hold:

$$w(x,0) = p(x), \;\; w(0,t) = u(t), \;\; w(L,t) = 0.$$

By discretizing in space we get

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \mathbf{F}(\mathbf{w},\mathbf{u}) + \mathbf{B}_1\mathbf{w}\mathbf{u} + \mathbf{B}_0\mathbf{u} = \mathbf{0} \quad \text{where} \quad \mathbf{F}(\mathbf{w},\mathbf{u}) = (\mathbf{A}_1\mathbf{w}) * \mathbf{w} \quad \text{and}$$

$$\mathbf{A} = \frac{\nu}{h^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}, \; \mathbf{A}_1 = \frac{-1}{2h}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$\mathbf{B}_1 = \frac{1}{2h}\mathbf{e}_1\mathbf{e}_1^*, \;\; \mathbf{B}_0 = \frac{\nu}{h^2}\mathbf{e}_1, \;\; \nu = 0.01.$

We can now write the nonlinearity in terms of a Kronecker product:

$\mathbf{F}(\mathbf{w},\mathbf{u}) = \hat{\mathbf{A}}_1\,(\mathbf{w}\otimes\mathbf{w})$, where $\hat{\mathbf{A}}_1 \in \mathbb{R}^{n\times n^2}$ and $\mathbf{w}\otimes\mathbf{w} \in \mathbb{R}^{n^2}$.

Carleman bilinearization

Next we make use of the **Carleman bilinearization**. For this purpose we introduce the new variable $\mathbf{x} = \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \otimes \mathbf{w} \end{bmatrix}$. Then the equation above leads to a **bilinear** system:

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} \mathbf{A}_1 & \frac{1}{2}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_1 \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}_0 & \mathbf{0} \end{bmatrix}}_{\mathbf{N}} \mathbf{x}\mathbf{u} + \underbrace{\begin{bmatrix} \mathbf{B}_0 \\ \mathbf{0} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

The output is the average of w, i.e. $\mathbf{y} = \underbrace{\frac{1}{n}[1, \cdots, 1, 0, \cdots, 0]}_{\mathbf{C}} \mathbf{x}$. Thus we obtain the bilinear system

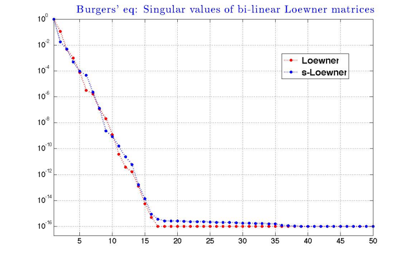
$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}} \qquad \text{where} \qquad \boxed{\mathbf{A},\ \mathbf{N} \in \mathbb{R}^{n^2 \times n^2},\ \ \mathbf{B},\ \mathbf{C}^* \in \mathbb{R}^{n^2}}$$

Reduction methods:

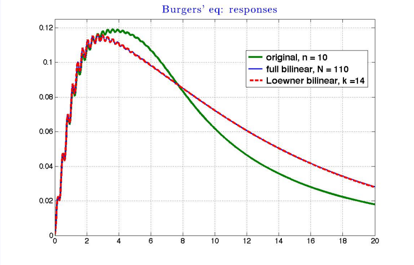
Carleman bilinearization	→ Loewner reduction
	→ Bilinear IRKA
Quadratic system	→ Loewner reduction

Dimensions: nonlinear: 10, bilinear 110, Loewner reduced 14 (based on 100 measurements).

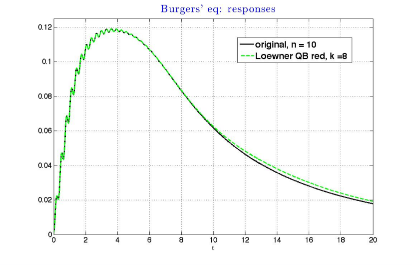
Burgers eq: singular values of $\mathbb{L}$, $\mathbb{L}_\sigma$



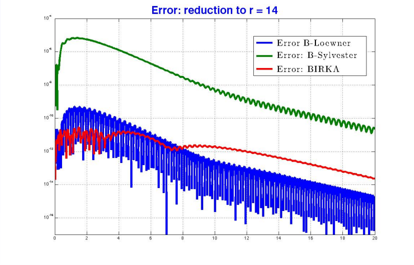
Time-domain response: decaying exponential



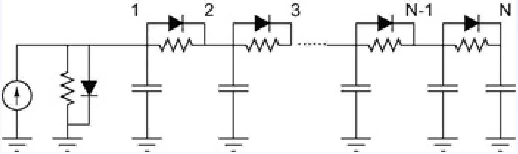
Time-domain response: decaying exponential



Error curves



Nonlinear RC circuit (Gu, Benner-Breiten)



$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{b}u(t),\ \mathbf{y}(t) = \mathbf{b}^*\mathbf{x}(t), \quad \text{where}$$

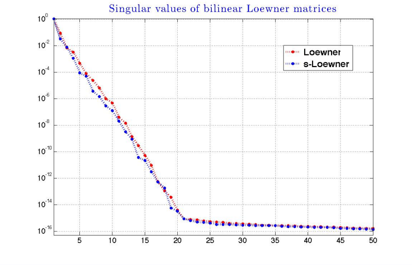
$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -g(\mathbf{x}_1) - \mathbf{x}(\mathbf{x}_1 - \mathbf{x}_2) \\ g(\mathbf{x}_1 - \mathbf{x}_2) - \mathbf{x}(\mathbf{x}_2 - \mathbf{x}_3) \\ \cdots \\ g(\mathbf{x}_{k-1} - \mathbf{x}_k) - \mathbf{x}(\mathbf{x}_k - \mathbf{x}_{k+1}) \\ \cdots \\ g(\mathbf{x}_{N-1} - \mathbf{x}_N) \end{bmatrix}$$

$$\mathbf{f}(\mathbf{x}) = \mathbf{e}^{40\mathbf{x}} + \mathbf{x} - 1,\ \ \mathbf{b}^* = [1,\ 0,\ \cdots,\ 0]$$

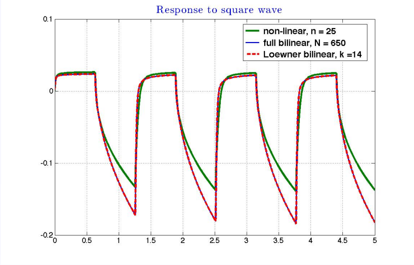
Reduction methods:

Carleman bilinearization	$\rightarrow$ Loewner reduction
	$\rightarrow$ Bilinear IRKA
Quadratic system	$\rightarrow$ Loewner reduction

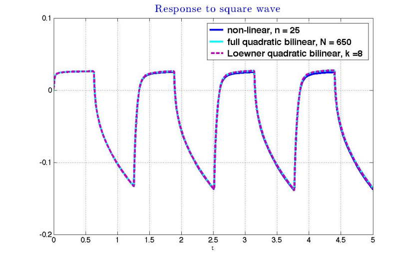
Nonlinear circuit: singular values of $\mathbb{L}$, $\mathbb{L}_\sigma$



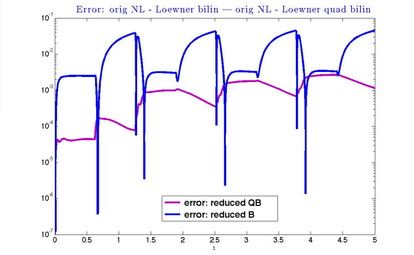
Time domain response to square pulse



Time domain response to square pulse



Error curves



The FitzHugh-Nagumo (F-N) system (Chaturantabut, Sorensen)

S. Chaturantabut, D. Sorensen, Nonlinear Model Reduction via Discrete Empirical Interpolation, SIAM Journal on Scientific Computing, 32, pp. 2737–2764, 2010.

F-N system: used in neuron modeling (simplified version of the Hodgkin-Huxley model) modeling activation and deactivation dynamics of a spiking neuron. Let $x \in [0, L]$, $t \geq 0$:

$$\begin{aligned} \epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) + c, \\ w_t(x,t) &= bv(x,t) - \gamma w(x,t) + c, \end{aligned}$$

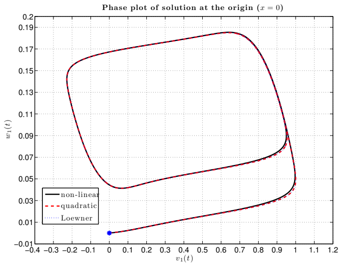
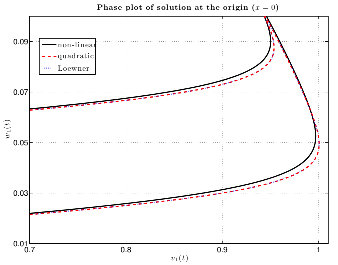
where

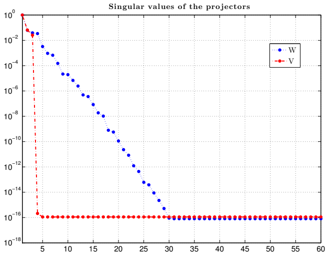
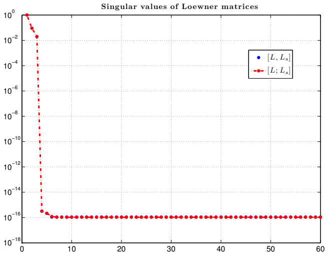
- $f(v) = v(v - 0.1)(1 - v)$
- initial and boundary conditions $v(x,0) = 0$, $w(x,0) = 0$, $x \in [0, L]$, $v_x(0,t) = -i_0(t)$, $v_x(L,t) = 0$, $t \geq 0$;
- the parameters are: $L = 1$, $\epsilon = 0.015$, $b = 0.5$, $\gamma = 2$, and $c = 0.05$.
- stimulus $i_0(t) = 50000\, t^3 exp(-15t)$.
- v and w: voltage and recovery of voltage.

The dimension of the full-order system (finite dfference) is 1024. The POD basis vectors are constructed from 100 snapshot solutions obtained from the solutions of the full-order FD system at equally spaced time steps. This can be re-written as a **quadratic system**:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_1\, \mathbf{x} + \mathbf{A}_2\, \mathrm{kron}(\mathbf{x}, \mathbf{x}) + \mathbf{A}_3\, \mathrm{kron}(\mathrm{kron}(\mathbf{x}, \mathbf{x}),\, \mathbf{x}) + \mathbf{B}_1\, \mathbf{u}_1 + \mathbf{B}_2\, \mathbf{u}_2$$

Method: only quadratic systems and reduction using Loewner works.

Phase space plot of **v** vs **w**	Detail
	

Singular values of the projectors	Singular values of the Loewner matrices
	

Outline

Model reduction from data, measured or computed: Summary

- Given input/output data, we can construct with no computation, a singular high order model in generalized state space form.
- Key tool: Loewner pencil.
- In applications the singular pencil must be reduced at some stage.
- Natural way to construct full and reduced models:

  $\Rightarrow$ does not **force** inversion of **E**,

  $\Rightarrow$ can deal with many input/output ports.
- In this framework the singular values of $\mathbb{L}$, $\mathbb{L}_\sigma$ offer a trade-off between accuracy of fit and complexity of the reduced system.
- Extension to parametrized systems.
- Extension to nonlinear systems by means of generalized Loewner and shifted Loewner matrices.
- Philosophy: $\boxed{\text{Collect data and extract desired information}}$

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Patent partly based on the Loewner framework

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(54) **STATE-SPACE MODEL-BASED SIMULATORS AND METHODS**

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