On Capacity of Deterministic Wireless Networks under Node Half-duplexity Constraint

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***Abstract*— In this paper we study the maximum throughput of a single multicast session in wireless networks under a practical constraint that the nodes are half-duplex, i.e. they cannot transmit and receive at the same time. We adapt "deterministic channel model" proposed recently in [1] for modeling wireless channel interference. The problem of finding the throughput capacity of a single multicast session has been studied analytically based on this channel model in [2]–[4]; a cutset bound found and its achievability is proven under the assumption that the wireless nodes are full-duplex. Here, we derive a new cutset bound on the maximum multicast throughput of half-duplex networks and propose a network coding scheme to achieve the cutset bound.**

I. Introduction

In recent years, network coding has become an important research topic in network information theory. It has been shown that network coding can help to improve the throughput, energy consumption, delay, robustness, security, and some other performance metrics of communication networks (see [5] to learn about various applications). Network coding was first introduced in the seminal paper by Ahlswede et al. [6] in which it was proven that the maximum flow capacity of a single multicast session can be achieved using network coding in wired networks with directional links. Later, [7] and [8] showed constructively that the linear network codes can achieve the minimum cutset bound of a single multicast session as well. Several network coding schemes for multicasting have been proposed more recently; some important studies can be found in [9].

Recently, a deterministic approach to study wireless networks was introduced in [1]. This model incorporates both broadcast and interference challenges in the wireless network. However, by removing the randomness this model makes the challenging problem of network coding for wireless network analytically tractable. For example, the problem of maximum flow capacity of a single multicast session in wireless networks was studied in [2] where it was shown that similar to the results for wired networks [6], the minimum cutset bound can be achieved. More recently, some random [3] and deterministic [4] linear network coding schemes have been proposed for achieving the multicast cutset bound. However, notice that the mentioned papers assume full-duplexity for the wireless nodes to study the network throughput.

In this paper we study the maximum throughput of a single multicast session under a practical assumption that the wireless nodes are half-duplex, i.e. they cannot transmit and receive simultaneously. We adapt "deterministic channel model" [1] for modeling wireless channel. In half-duplex networks a scheduling scheme is needed to transport information bits throughout the network. Given the scheduling scheme, we derive a cutset bound on the throughput capacity of a multicast session. Then, we construct a coding scheme which established upon the existing network codes in [2]–[4] (proposed for full-duplex model) to achieve the half-duplex cutset bound. Based on the knowledge of authors, this is the first work that studies the throughput capacity of a single multicast session in general half-duplex wireless networks.

This paper is organized as follows. In Section II, we describe the network model and notations. We explain the multicast cutset bound and our proposed network coding scheme in Section III. Finally, we conclude the paper in Section IV.

II. Network Model and Notations

We consider a wireless network as a directed graph where each node can transmit the same message into *all* its outgoing links and receives the *superposition* of the signals arrived from the incoming links. We adopt the deterministic model in [1], [2] to model the gain of the links and how the superposition is performed. Note that in the most existing papers which apply this channel model assume that the nodes are *full-duplex*, i.e. the nodes simultaneously transmit and receive data. Therefore, they do not deal with the notion of scheduling on the transmissions of network nodes and its impact on the analytical results. However, in this paper we assume that the nodes are *half-duplex*. The half-duplexity assumption is close to reality, since the usual wireless devices have single-radio and work usually in either transmission or reception mode but not both. Similar to the existing work, a simplification assumption for considering the wireless links to be directional exists in our network model. This assumption can be valid when the links are set to be used only in one direction, also the nodes are able to tune on one of multiple available channels so that the links with different receiver nodes do not interfere with each other. Removing this assumption can extremely complicate the analysis, hence we leave it for the future studies.

We assume that the network contains $1 + N$ nodes, where one of them is the source of a single multicast session and the rest of the nodes are relay nodes or terminals (destinations) of the session. We use a universal index k for every node Φ_k where $k = 0, 1, \ldots, N$.

A. Deterministic Channel Model

In this channel model, the output signal from node Φ_k at time-slot t is considered as a column vector $\mathbf{y}_t^k = [y_{t,1}^k, y_{t,2}^k, \ldots, y_{t,q}^k]^\dagger$ of size q, where each element is a value in Galois Field $\mathbb{F}(p^n)$ for some prime number p and positive integer n. Here $\dagger$ is used to denote the matrix transpose operation. Each link from the node Φ_i to Φ_k in the network is denoted by its transfer function $\mathbf{G}_i^k$ which is a $q \times q$ matrix with the entries in $\mathbb{F}(p^n)$. The output of this link is equal to $\mathbf{G}_i^k \mathbf{y}_t^i$. The received vector or input at the node Φ_k is a column vector $\mathbf{x}_t^k = [x_{t,1}^k, x_{t,2}^k, \ldots, x_{t,q}^k]^\dagger$ which is the superposition of the outputs of the links arriving at node Φ_k defined on component-by-component basis, i.e.,

$$\mathbf{x}_t^k = \sum_{i=0}^{n} \mathbf{G}_i^k \mathbf{y}_t^i \tag{1}$$

where $\mathbf{G}_i^k$ is the transfer function when there is an outgoing link from Φ_i to Φ_k, otherwise it set to $q \times q$ matrix $\mathbf{0}$. Note that in the half-duplex model where the nodes have some scheduling for transmissions, if a node Φ_i does not transmit at time-slot t, we can either remove it from above sum or we set $\mathbf{y}_t^i$ to be $q \times 1$ vector $\mathbf{0}$.

If we stack together the received vectors at multiple nodes $\Phi_k, k \in \mathcal{B} = \{j_1, \ldots, j_b\}$, assuming they are characterized by the output at the nodes $\Phi_k, k \in \mathcal{A} = \{i_1, \ldots, i_a\}$ (i.e. all incoming links of $\mathcal{B}$ originated in $\mathcal{A}$), then the transfer function is given by

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_t^{j_1} \\ \mathbf{x}_t^{j_2} \\ \vdots \\ \mathbf{x}_t^{j_b} \end{bmatrix} &= \begin{bmatrix} \mathbf{G}_{i_1}^{j_1} & \mathbf{G}_{i_2}^{j_1} & \ldots & \mathbf{G}_{i_a}^{j_1} \\ \mathbf{G}_{i_1}^{j_2} & \mathbf{G}_{i_2}^{j_2} & \ldots & \mathbf{G}_{i_a}^{j_2} \\ \vdots & & & \vdots \\ \mathbf{G}_{i_1}^{j_b} & \mathbf{G}_{i_2}^{j_b} & \ldots & \mathbf{G}_{i_a}^{j_b} \end{bmatrix} \begin{bmatrix} \mathbf{y}_t^{i_1} \\ \mathbf{y}_t^{i_2} \\ \vdots \\ \mathbf{y}_t^{i_a} \end{bmatrix} \\ &= \mathbf{G}_{\mathcal{A}}^{\mathcal{B}} \begin{bmatrix} \mathbf{y}_t^{i_1} \\ \mathbf{y}_t^{i_2} \\ \vdots \\ \mathbf{y}_t^{i_a} \end{bmatrix} \end{aligned} \tag{2}$$

The above notation of $\mathbf{G}_{\mathcal{A}}^{\mathcal{B}}$ will be used in several places again in the paper.

Notice that a vector $\mathbf{x} \in (\mathbb{F}(p^n))^q$ can include at most $n \log_2(p)$ information bits. Therefore, the rate of information that can be sent from $\mathcal{A}$ to $\mathcal{B}$ is at most $n \log_2(p) |\mathbf{G}_{\mathcal{A}}^{\mathcal{B}}|$. Here, $|.|$ denotes the matrix rank.

III. The Cutset Bound and Its Achievability in Half-duplex Wireless Networks

In this section we study the maximum throughput of a single multicast session in wireless networks. In half-duplex network model, a scheduling scheme for the transmissions (and receptions) of the nodes is needed in order to transport information in the network. Obviously, the throughput of the network depends on the scheduling scheme in addition to other factors which affect the throughput in full-duplex model such as the network topology and the transfer functions of the links.

Before starting our analytical framework on the half-duplex model, we present two examples to highlight some fundamental differences on throughput analysis between half-duplex model and full-duplex model.

First example shows that finding the minimum cutset bound in half-duplex model even for a single source and destination pair is very complicated in comparison with full-duplex model. Fig. 1 shows a network with two paths from a source (Φ_0) to a destination (Φ_4).

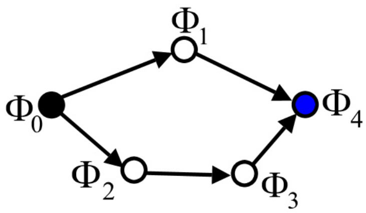

Fig. 1. An example for a half-duplex network the cutset bound from Φ_0 and Φ_4 on each of two paths $\frac{1}{2}qn\log_2(p)$; while the minimum cutset bound of these paths together is bounded by $\frac{5}{6}qn\log_2(p)$.

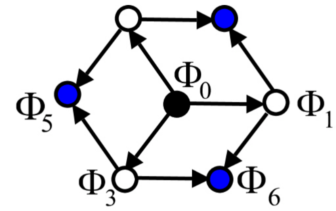

Fig. 2. An example for a half-duplex network where the minimum cutset for a each of terminals $\{\Phi_4, \Phi_5, \Phi_6\}$ separately is $qn\log_2(p)$; while the minimum cutset of the multicast session toward these terminals is $\frac{5}{6}qn\log_2(p)$.

We assume that the transfer functions of all links are equal to $\mathbf{I}$. It is easy to see that the maximum rate of sending information over each path will be half of full-duplex case, i.e. $\frac{1}{2}qn\log_2(p)$, because, at any time instance Φ_1 and Φ_2 are either either in transmission or reception modes but not both. However, note that we cannot manage to use both paths at the rate of $\frac{1}{2}qn\log_2(p)$ simultaneously. In fact, we can show that the maximum rate (cutset bound for half-duplex model) is equal to $\frac{5}{6}qn\log_2(p)$.

To prove the above statement, we denote the fractions of time that links $\Phi_0\Phi_2$ and $\Phi_3\Phi_4$ work (bits are transmitted through them) together by α. Also, the fraction of time that these links do not work simultaneously by β. This means that the rate of information sent through $\Phi_0\Phi_2\Phi_3\Phi_4$ is $\alpha+\beta$.

Note that $\Phi_2\Phi_3$ cannot work simultaneously $\Phi_0\Phi_2$ and $\Phi_3\Phi_3$. Then, considering the scheduling constraints on the links of path $\Phi_0\Phi_2\Phi_3\Phi_4$, we have

$$2\beta + \alpha + (\alpha+\beta) \leq 1 \tag{3}$$

The maximum rate of information that can be sent through $\Phi_0\Phi_1\Phi_4$ is $(1-\alpha)/2$. Then, the maximum rate R will be bounded as

$$R \leq qn\log_2(p)[\alpha+\beta+(1-\alpha)/2] \tag{4}$$

Taking inequality (3) into account, we have $R \leq qn\log_2(p)[(1/2+(2\alpha+3\beta)/3-\alpha/6] \leq \frac{5}{6}qn\log_2(p)$.

This cutset bound can be achieved when $\alpha = 0$ and $\beta = 1/3$. Then, $\frac{1}{2}qn\log_2(p)$ bit per second is sent through $\Phi_0 \to \Phi_1 \to \Phi_4$ and $\frac{1}{3}qn\log_2(p)$ bit per second is sent through the other path.

This simple example shows that finding the exact cutset bound of half-duplex network can be very complicated when the number of nodes and paths grow or the transfer function of the links are defined non-trivially.

In the second example we explain a fundamental difference between full-duplex and half-duplex model which is about the throughput of multicast sessions. Fig. 2 shows a network with a multicast session from a source Φ_0 to three terminals Φ_4, Φ_5, Φ_6. Assume that the transfer functions of all links are equal to $\mathbf{I}$. It is easy to show that the source node Φ_0 can unicast information to each terminal at the rate of $qn\log_2(p)$; for achieving this rate the links of two paths from the source toward the terminal must be used in opposite order. However, note that we are not able to multicast information to all three terminals at this rate. In fact we can prove that the maximum throughput of this multicast session is bounded by $\frac{5}{6}qn\log_2(p)$. For proving this, we show that two of relay nodes Φ_1, Φ_2, Φ_3 have to be simultaneously in transmission or reception mode for at least $1/3$ fraction of time, so the terminal which is connected to those two loses half of $1/3$ fraction of the time for obtaining new information. Therefore, the receiving rate of this terminal (also the maximum multicast throughput) will be bounded by $(1-\frac{1}{6})qn\log_2(p)$.

This example shows that in contrast to full-duplex model we cannot compute the cutset bound of multicast session by only considering the cutset bound from the source to each terminal separately. In half-duplex model the cutset bound of a multicast session is computed only by finding the best scheduling scheme for sending the information to all terminals simultaneously.

Now we demonstrate our analytical framework for finding the throughput capacity of a multicast session. An interesting method for computing a cutset bound on the throughput of half-duplex wireless network has been proposed in [10]. The idea of the paper is to consider every set of simultaneous transmissions $\Theta_s \subseteq \{\Phi_0, \Phi_1, \ldots, \Phi_N\}$ as a state for the network. Clearly, the number of states is equal to the number of subsets of the nodes which is 2^{N+1}.

Any arbitrary scheduling scheme that transports the information between some nodes can be considered as a set of states of the network where each state is used in some particular periods of time. For any given scheduling scheme, *time-proportion* parameters $\tau_1, \ldots, \tau_{2^{N+1}}$ are defined as the fraction of time where the scheduling scheme is at state $\Theta_1, \ldots, \Theta_{2^{N+1}}$ respectively. To be mathematically rigorous, we define the time-proportion $\tau_s = \lim_{T\to\infty} \frac{T_s}{T}$ for $s = 1, \ldots, 2^{N+1}$ where T_s is the total duration of time in time interval $[0, T]$ that the network is at state s. Note that the time-proportions can be defined only for those scheduling schemes where the above limit converges. It is straightforward to show that convergence exists for any periodic scheduling scheme.

In [10] a cutset bound is found on the time-proportions for any given scheduling scheme. In Lemma 1, we adapt Theorem 2 of [10] under deterministic channel model explained in Section II.

Lemma 1: Assume a scheduling scheme with time proportions $\tau_1, \ldots, \tau_{2^{N+1}}$. The rate of information (R) that can be sent from a node Φ_0 to a node Φ_{k_i} is bounded as

$$R \leq \min_{\Omega\in\Lambda_{k_i}} \{ \sum_{s=1}^{2^{N+1}} \tau_s \cdot n \log_2(p) |\mathbf{G}_{\Omega\cap\Theta_s}^{\Omega^c\cap\Theta_s^c}| \} \qquad (5)$$

where Λ_{k_i} denotes the sets of cutsets between Φ_0 and Φ_{k_i}, also Ω^c and Θ^c are the complement set of Ω and Θ respectively.

Note that in (5), $n \log_2(p) |\mathbf{G}_{\Omega\cap\Theta_s}^{\Omega^c\cap\Theta_s^c}|$ is the maximum rate of information which can be sent from the set Ω to Ω^c assuming that the network is at state s where the nodes of Θ_s are in transmission mode and the nodes of Θ_s^c are in reception mode.

We can slightly change the equation (5) to generalize the computed cutset bound for multicast case. Here, we find the minimum cutset among all terminals for any given scheduling scheme.

Lemma 2: Assume a scheduling schemes with time proportions $\tau_1, \ldots, \tau_{2^{N+1}}$. Consider a multicast session from a source node Φ_0 to a set of terminals $\Gamma = \{k_1, \ldots, k_d\}$. The maximum rate of information which can be sent from the source to all terminals (R) is bounded as

$$R \leq \min_{\Phi_{k_i}\in\Gamma} [\min_{\Omega\in\Lambda_{k_i}} \sum_{s=1}^{2^{N+1}} \tau_s \cdot n \log_2(p) |\mathbf{G}_{\Omega\cap\Theta_s}^{\Omega^c\cap\Theta_s^c}|] \quad (6)$$

As we explained in the second example, in half-duplex network model, the same values of $\tau_1, \ldots, \tau_{2^{N+1}}$ might not maximize (5) when we vary the terminal node Φ_{k_i} in the network. For finding the throughput capacity the time fractions must be chosen such that the right side of equation (6) is maximized.

In Theorem 3 we assume that a scheduling scheme is given for the half-duplex network, then we prove that the the minimum cutset bound computed in Lemma 2 in can be achieved in half-duplex acyclic wireless networks using a proper network coding scheme.

Theorem 3 (Achievability Theorem for Half-duplex): Assume a single multicast session in a half-duplex acyclic wireless network with a given a scheduling scheme. Then there exists a coding scheme to achieve the minimum cutset bound of the multicast session given in equation (6).

Proof of Theorem 3: We consider a a large integer number M. We define $n_s = \lfloor \tau_s M \rfloor$ for $s = 1, 2, \ldots, 2^{N+1}$. Clearly, for large M we have $\tau_s \simeq n_s/M$ (more precisely $n_s/M < \tau_s < n_s/M + 1/M$).

Next, we define Large Time Scale (LTS) network model for the half-duplex network as following: (i) Each LTS time-slot is equal to M uses of wireless channel. (ii) LTS model follows the ”deterministic channel model”, however the size of input and output vectors for each LTS slot is qM. (iii) We denote the LTS input vector of Φ_i and LTS output vector of Φ_j by $\bar{\mathbf{x}}_i$ and $\bar{\mathbf{y}}_j$ respectively. The LTS vectors contain M usual input or output vectors (of size q) where n_s of them correspond to the input and output of the nodes at state s for $s = 1, 2, \ldots, 2^{N+1}$ (note that $\sum n_s \leq M$). (iv) The LTS transfer function between nodes Φ_i and Φ_j denoted by $\bar{\mathbf{G}}_{ij}$ which is a $qM \times qM$ matrix. Similar to the structure the LTS vectors, $\bar{\mathbf{G}}_{ij}$ has M submatrices of size $q \times q$ on its diagonal where n_s submatrices correspond to the transfer function between node Φ_i and node Φ_j at state s that is equal to $\mathbf{G}_{ij}$ if and only if $\Phi_i \in \Theta_s$ and $\Phi_j \in \Theta_s^c$. The rest of entries of $\bar{\mathbf{G}}_{ij}$ are equal to zero,

Now, we demonstrate the LTS model of the half-duplex network can be seen analytically as a full-duplex network. Every node stores all information that it receives in one LTS slot and it processes the information and transmits in the next LTS time-slots. Similar to full-duplex model there is at least one time-slot difference between what a node receives and transmits.

It is straightforward to show that the cutset rank of LTS network for an arbitrary set Ω is equal to $|\bar{\mathbf{G}}_{\Omega}^{\Omega^c}| = \sum_{s=1}^{2^{N+1}} n_s |\mathbf{G}_{\Omega\cap\Theta_s}^{\Omega^c\cap\Theta_s^c}|$. The minimum cutset of LTS network is the minimum cutset bound among all terminals that is computed as following

$$\bar{r}_{\min} = \min_{\Phi_{k_i} \in \Gamma} \min_{\Omega \in \Lambda_{k_i}} \sum_{s=1}^{2^{N+1}} n_s |\mathbf{G}_{\Omega \cap \Theta_s}^{\Omega^c \cap \Theta_s^c}| \tag{7}$$

The existing work on full-duplex model show that there exists a network coding scheme which achieves throughput of $n\log_2(p) \cdot \bar{r}_{\min}$ [2]–[4].

Since each LTS time-slot corresponds to M channel uses, therefore the rate which is sent in the original network can be found by dividing $n\log_2(p)\bar{r}_{\min}$ to M. Thus,

$$\begin{aligned} R &= n\log_2(p)\bar{r}_{\min}/M \\ &= n\log_2(p) \min_{\Phi_{k_i} \in \Gamma} \min_{\Omega \in \Lambda_{k_i}} \sum_{s=1}^{2^{N+1}} n_s/M \cdot |\mathbf{G}_{\Omega \cap \Theta_s}^{\Omega^c \cap \Theta_s^c}| \\ &\simeq n\log_2(p) \min_{\Phi_{k_i} \in \Gamma} \min_{\Omega \in \Lambda_{k_i}} \sum_{s=1}^{2^{N+1}} \tau_s |\mathbf{G}_{\Omega \cap \Theta_s}^{\Omega^c \cap \Theta_s^c}| \qquad (8) \end{aligned}$$

Clearly, achievable rate R becomes equal to (6) as $M \to \infty$. ■

In Corollary 4 we compute the maximum throughput of a single multicast session in an acyclic wireless network. The bound can be found by optimizing the time proportions of different states and then computing the minimum cutset bound. As we explained in Theorem 3 the minimum cutset bound of any scheduling scheme can be achieved using a network coding scheme.

Corollary 4: The maximum rate of information which can be sent from the source of a single multicast session Φ_0 to a set of terminals $\Gamma = \{\Phi_{k_1}, \dots, \Phi_{k_d}\}$ in an acyclic wireless network is

$$R = n\log_2(p) \max_{\sum \tau_s = 1} \{ \min_{\Phi_{k_i} \in \Gamma} [\min_{\Omega \in \Lambda_{k_i}} \sum_{s=1}^{2^{N+1}} \tau_s |\mathbf{G}_{\Omega \cap \Theta_s}^{\Omega^c \cap \Theta_s^c}|] \} \tag{9}$$

where τ_s is the fraction of time that the network is at state Θ_s, and Λ_{k_i} denotes the set of cutsets between Φ_0 and Φ_{k_i}.

IV. Conclusion

In this paper we studied the throughput capacity of a multicast session in wireless networks with constraint of half-duplexity for the nodes. We adapted deterministic channel model for modeling wireless interference in the network. We highlighted some fundamental differences for computing cutset bounds between the former full-duplex model and the half-duplex model. Next, we drove a new cutset bound on the maximum throughput of a multicast session with a given scheduling scheme. Then, we proved the achievability for the cutset bound using a network coding scheme.

Future work concerns finding an optimal or near-optimal scheduling technique for the half-duplex network to achieve the throughput capacity of a multicast session given in (9).

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