Bounds for the Capacity of Wireless Multihop Networks Imposed by Topology and Demand

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ABSTRACT

Existing work on the capacity of wireless networks predominantly considers homogeneous random networks with random work load. The most relevant bounds on the network capacity, e.g., take into account only the number of nodes and the area of the network. However, these bounds can significantly overestimate the achievable capacity in real world situations where network topology or traffic patterns often deviate from these simplistic assumptions. To provide analytically tractable yet asymptotically tight approximations of network capacity we propose a novel *space-based approach*. At the heart of our methodology lie simple functions which indicate the presence of active transmissions near any given location in the network and which constitute a tool well suited to untangle the interactions of simultaneous transmissions. We are able to provide capacity bounds which are tighter than the traditional ones and which involve topology and traffic patterns explicitly, e.g., through the length of Euclidean Minimum Spanning Tree, or through traffic demands between clusters of nodes. As an additional novelty our results cover unicast, multicast and broadcast and are asymptotically tight. Notably, our capacity bounds are simple enough to require only knowledge of node location, and there is no need for solving or optimizing multi-variable equations in our approach.

Categories and Subject Descriptors

Computer Systems Organization [**Computer- Communication Network**]: Network Architecture and Design Wireless communication; Data [**Coding and Information Theory**]: Formal models of communication

General Terms

Design, Performance, Theory

Keywords

Capacity of Wireless Networks, Transport Capacity

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1. INTRODUCTION

There has been a growing interest to understand the fundamental capacity limits of wireless networks [11, 1, 6, 8, 18, 23, 17, 22, 24, 13]. Results on network capacity are not only important from a theoretical point of view but also provide guidelines for protocol design in wireless networks. Hitherto, most research on network capacity has focused on the capacity of random homogeneous networks with connections between randomly selected nodes [11, 6, 15, 23].

In this paper, we study static wireless networks with the goal of assessing the impact of topology and traffic pattern on capacity. Our methodology takes an approach based in the very space of the network thereby avoiding graph-theoretical arguments and computations. At the heart of our method lies the concept of an transmission *arena*, a domain defined for each transmission which at the same time is simple yet represents sufficiently faithfully the part of the space which is affected by the transmission. To give the story in a nutshell, for well chosen arenas the underlying channel model imposes restrictions on simultaneous transmissions in terms of an *arena-bound*. These restrictions are then exploited in several ways to provide novel capacity-related bounds.

As the first contribution of this paper, we introduce the novel concept of arenas and demonstrate how to compute the arena-bound for three well known wireless channel models (Protocol Model, Physical Model and Generalized Physical Model [11, 1]).

As our second set of contributions, we study the transport capacity C_T of arbitrary wireless networks as introduced by Gupta and Kumar [11]. We compute a novel upper bound for C_T in terms of a simple topological quantity, the length of Euclidean Minimum Spanning Tree (EMST) of the network. This is achieved by averaging the arena-bound over appropriate curves. Applying well known bounds on the length of EMST, one recovers the traditional bounds on the transport capacity [11, 10] as a special case of our results. Interestingly, we are also able to establish the tightness of the upper bound based on the EMST. Furthermore, for the popular cluster topology, we obtain an upper bound on C_T in terms of the number of clusters and the clusters radius using the EMST.

As our third contribution, we address the throughput of simultaneous flows by averaging the arena-bound over time. We demonstrate the effectiveness of the method be solving concrete problems: we find an upper bound for the capacity region of the rates of single path flows, as well as an upper bound for the maximum flow rate between two parts of the network.

As our fourth contribution, we derive capacity bounds for more general applications such as multicast. This is achieved by averaging the arena-bound appropriately over space and time.

The paper is organized as follows. In Section 2 we summarize existing works on the network capacity. We introduce a wireless network and channel models in Section 3. In Section 4 we introduce the concept of transmission arenas and demonstrate the relevant properties under the three mentioned channel models. In Section 5 we compute novel topology-based bounds for the transport capacity. In Section 6 we bound the throughput of certain flows. In Section 7 we address multicast applications. Finally, we conclude the paper in Section 8. All proofs are placed in the Appendix.

2. RELATED WORK

Gupta and Kumar [11] study the network capacity for unicast connections in static wireless networks consisting of n nodes distributed in a circle of area A with wireless channel capacity W. They define the "transport capacity" of a wireless network with units of bit-meters per second as the maximum rate of the packets times the distance they travel between the source and the destination. Their main result says that the aggregate transport capacity of unicast connections is $O(W\sqrt{An})$ in an arbitrary and it is $\Theta(W\sqrt{An/\log(n)})$ in a random network where the nodes are placed uniformly. As a result, if the capacity is shared between random sources and destinations in the network, per node capacity decreases as $O(W\sqrt{1/n})$ (in random networks $\Theta(W\sqrt{1/n\log(n)})$) when n grows. The same authors also prove that if the nodes are distributed in a sphere with volume V then the aggregate transport capacity is $O(W\sqrt[3]{Vn^2})$ [10]. Later, these results were generalized for a more accurate channel model in [1].

Since the seminal work of Gupta and Kumar, the network capacity problem has been studied using different approaches. For example [6] introduces unicast schemes for random source and destination nodes in large random homogeneous network which achieve $\Theta(W\sqrt{1/n})$ per node capacity by applying percolation theory results. Other work computes the capacity of networks with multiple channels [16] and with ultra bandwidth channel [18].

For wireless mobile networks, Grossglauser and Tse [8] show that per node capacity can be increased to $\Theta(W)$ if packet delay is left unbounded. They propose a mobility-based routing method in which the number of retransmissions of the unicast packets between source and destination is reduced to two. Many other efforts demonstrate that there is a trade-off between the capacity and the delay in wireless mobile networks, for different mobility patterns and constraints on delay (see [21] for references).

Note that all the mentioned papers consider a grid type network topology (in random networks, the nodes are distributed homogeneously) with symmetric traffic pattern (the traffic is distributed among the node uniformly) for proving the achievability of the computed upper bounds of [11, 1]. However, for a different network topology or traffic pattern the network capacity could become significantly less than these upper bounds. Indeed, in the analysis of [11, 1] the effect of topology and traffic pattern are ignored and the computed upper bounds are only in terms of number of nodes and the area of the network.

Introducing a new direction in network capacity research, this paper goes well beyond this existing work, taking network topology and traffic pattern into account.

It should also be mentioned that there exists work on the capacity of wireless networks for multicast and broadcast applications. Some asymptotic capacity bounds computed in random networks with symmetric traffic in [23, 22, 24]. More recently, [13, 14] computes the broadcast capacity in arbitrary wireless networks. Interestingly, the framework of this paper can be generalized and applied for computing the capacity of arbitrary networks for multicasting and broadcasting. In fact, our results cover the traditional bounds on unicast, multicast and broadcast capacity.

Note that all the above mentioned papers as well as this paper assume only point-to-point coding at the receivers. If the nodes are allowed to use sophisticated multi-user coding then a per-node capacity of a higher order than that described above can be achieved [12, 7, 9, 19]. A full discussion of these results is beyond the scope of this paper.

3. WIRELESS CHANNEL MODELS AND BASIC NOTIONS

In this section, we describe the models and notions used in this paper. We consider a wireless network consisting of n wireless nodes in d-dimensional space ($\mathbb{R}^d$). We denote the set of transmitter-receiver pairs of *simultaneous direct* transmissions active at time τ by $\mathcal{SD} := \{(S_1, D_1), (S_2, D_2), \ldots, (S_m, D_m)\}$. Also, we denote the set of transmitters by $\mathcal{S} := \{S_1, \ldots, S_m\}$. Note that these sets vary over time; if not otherwise indicated, however, we will consider one fixed but arbitrary time instant. For simplicity in notation, the node symbols are used also to represent their locations. For example, $|S_i - D_i|$ is the distance between the nodes S_i and D_i in $\mathbb{R}^d$.

3.1 Wireless Channel Models

This paper covers all of the common channel models found in the literature on wireless network capacity, namely the following three groups of models. First, the *Protocol Model* models a successful transmission based on the distance with the closest interfering transmitter [11, 10, 15, 2]. This model is the simplest of the three and easiest to analyze. Second, the *Physical Model* sets a threshold on the *Signal to Interference plus Noise Ratio* (SINR) of the received signal, declaring the transmission to be successful if the SINR is larger than the threshold [11, 10, 1]. Third, the *Generalized Physical Model* determines the transmission rate in terms of the SINR [1, 6, 22] by using Shannon's capacity formula for a wireless channel with additive Gaussian white noise [4].

3.1.1 Protocol and Physical Model

In both, the Protocol and the Physical Model the assigned transmission rate from node $S_i \in \mathcal{S}$ to node D_i is modelled as

$$W_i = \begin{cases} W & \text{if successful} \\ 0 & \text{if unsuccessful or inactive.} \end{cases} \quad (1)$$

where W, is the *channel capacity*.

What distinguishes the models are the conditions for a transmission to be modelled as successful. In the literature [11, 2, 10] one finds the following three different versions under the term “Protocol Model”. Given the *interference parameter* $\Delta > 0$ a transmission is modelled as successful if:

- (Protocol Model 1):
  $|S_k - D_i| \geq (1+\Delta)|S_k - D_k|$ for all $S_k \in \mathcal{S}\backslash\{S_i\}$.
- (Protocol Model 2):
  $|S_k - D_i| \geq (1+\Delta)|S_i - D_i|$ for all $S_k \in \mathcal{S}\backslash\{S_i\}$.
- (Protocol Model 3):
  $|S_k - D_i| \geq (1+\Delta)r$ for all $S_k \in \mathcal{S}\backslash\{S_i\}$, and $|S_i - D_i| \leq r$ where the *transmission range* r is an additional parameter.

Under the Physical Model a transmission is modelled as successful if

$$\mathrm{SINR} = \frac{P_i G_{ii}}{N + \sum_{k \neq i, k \in \mathcal{S}} P_k G_{ki}} \geq \beta \qquad (2)$$

Here, β is the SINR-threshold, N represents the ambient noise, and G_{ki} denotes the signal loss, meaning that $P_k G_{ki}$ is the receiving power at node D_i from transmitter S_k. We assume a low power decay for the signal loss of the form $G_{ki} = |S_k - D_i|^{-\alpha}$, where $\alpha > 0$ is the signal loss exponent. Throughout the paper we assume only $\alpha > 0$, except for Theorem 3, where $\alpha > d$.

3.1.2 Generalized Physical Model

In this model all node pairs are able to communicate by direct transmission, however with a rate W_i that depends on SINR as

$$W_i = B \log_2 \left(1 + \frac{P_i G_{ii}}{BN_0 + \sum_{k \neq i, k \in \mathcal{S}} P_k G_{ki}}\right) \qquad (3)$$

Here, B is the bandwidth of the wireless channel and $N_0/2$ is the noise spectral density. While this model assigns a more realistic transmission rate at large distance than the other two channel models, it also results in a singularity under the signal loss model $G_{ii} = |S_i - D_i|^{-\alpha}$: according to (3) the receiving power and the rate are amplified to unrealistic levels if transmitter and receiver are placed very closely to each other. The singularity can be easily addressed by upper bounding the received power at each node. Some papers have pointed out this [5, 3] and suggest a “bounded propagation model” for the rate. We do not study this version of the model in this paper, since its analysis can be performed in a straightforward way by using similar methods as put forward here for the two Physical Models.

3.2 Transport Capacity

The transport capacity is useful to study the transmissions for a given set of unicast source-destination pairs $\mathcal{UV} := \{(U_1, V_1), ..., (U_m, V_m)\}$. It can be defined as [11]:

$$C_T(\mathcal{UV}) := \max_{\text{multi-hop paths}} \sum_k |U_k - V_k| R_k \qquad (4)$$

where R_k is the average rate of unicast connection between of U_k and V_k over a given multi-hop path. The maximum is taken over all possible multi-hop routes establishing the required connections between the sources and destinations. A simple upper bound which actually does not depend on the set $\mathcal{UV}$ is found by noting that for the simultaneous routes achieving C_T there must be a time instance where the simultaneous direct hop-forwarding transmission reach at least C_T [11]:

$$C_T(\mathcal{UV}) \leq \max_{\mathcal{SD}} \sum_{(S_i, D_i) \in \mathcal{SD}} |S_i - D_i| W_i \qquad (5)$$

where the maximum is over all possible sets of simultaneous, direct transmissions $\mathcal{SD}$.

4. ARENAS: A SPATIAL FRAMEWORK

The problem of finding the optimal configuration for simultaneous transmissions and therefore the capacity of a network soon leads to forbiddingly complex and involved computations (NP hard, see [20]), especially when attempting an analytical solution.

To untangle the mutual interference of simultaneous transmissions and to achieve analytically tractable yet asymptotically tight approximations we move from the natural graph-based methodology to a *space-based approach.* In other words, rather than studying mutual restrictions and interference on the graph of nodes, we focus on restrictions regarding the *proximity* of ongoing transmissions as seen at *any location* X of the space occupied by the network.

4.1 Packing Simultaneous Transmissions

Consider a set of simultaneous successful transmissions under Protocol Model 3 in a planar network. It follows immediately from the definitions and the triangular inequality that the senders need to be at least at distance Δr from each other. Observing such restrictions between senders or receivers build natural ingredients of graph-theoretic approaches.

For our space-based approach consider an *arbitrary* point X in the plane, not necessarily a sender or receiver. Consider the senders in distance r from X; as noted the discs of radius $\Delta/2$ around these senders must be disjoint, yet they are contained in the disc of radius $(1 + \Delta/2)$ around X. Thus, there can be at the most $M := (1+2/\Delta)^2$ successful senders which include X in their radio range.

Intuitively, this tells us that under the Protocol Model 3 the rate of information which can be transmitted in the vicinity of any arbitrary point X is bounded by the given constant $M \cdot W$. In other words, the “local capacity” of the network or “packing density” of senders is bounded everywhere.

4.2 Unicast Transmission Arena

The above reasoning leads us to a new concept: A *transmission arena*, or arena for short, is any set A_i associated with a transmission of rate W_i (see Section 3) between sender S_i and receiver D_i satisfying the following two conditions. First, A_i is determined by S_i and D_i alone. Second, under the underlying channel model we have for any *fixed* time instance and any *point* $X \in \mathbb{R}^d$

$$\sum_{i \in \mathcal{SD}} W_i \cdot \mathbb{I}_{A_i}(X) \quad \leq \quad M \cdot W_o \qquad (6)$$

Here the so-called *arena-bound* M is allowed to depend only on the parameters of the channel model and the *number* of nodes, but not on node location or on the traffic patterns. Also,

$$W_o := \begin{cases} W & \text{Protocol and Physical Models,} \\ B & \text{Generalized Physical Model.} \end{cases} \qquad (7)$$

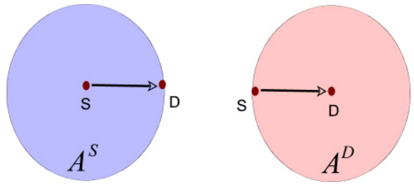

Figure 1: The disturbance area A^S of a transmitter and the protection area A^D of a receiver form arenas for most channel models (see Lamma 1).

From the motivating example of Section 4.1 we find that under the Protocol Model 3 the radio ranges around senders constitute arenas with arena-bound $M = (1 + 2/\Delta)^2$. Notably, this example reveals also that an arena-bound may be a simple consequence of the relative position of senders and/or destinations. Indeed, the potential of arenas lies less in capturing the relation between senders and rather in providing a *spatial framework* which compactly captures the interactions between simultaneous direct transmissions via (6) and their impact on *any* location X. Moreover, arenas are free of detailed information on topology or traffic patterns, and, most importantly, allow the use of integration and other analytical tools.

It is convenient to introduce the *arena-rate function*[1]

$$\phi_i(X) := W_i \cdot \mathbb{I}_{A_i}(X) \tag{8}$$

Again, the arena-rate function does not approximate the signal strength at X but rather provides a weighted indication of the *presence* of transmissions nearby X.

4.3 Arenas for the Classical Channel Models

An arena can be thought of as representing to a fair degree the interference caused at the sender, respectively the low noise level required at the receiver. Natural choices are:

$$\textit{disturbance area} \qquad A_i^S := \{X : |X - S_i| \le l_i\} \tag{9}$$

$$\textit{protection area} \qquad A_i^D := \{X : |X - D_i| \le l_i\} \tag{10}$$

where $l_i = |S_i - D_i|$ (see Figure 1). Extending the motivating argumentation 4.1 above, we are able to show that both form arenas with the same arena-bound under all models of Section 3 except Protocol Model 2, resp. Protocol Model 1.

Lemma 1. *The following sets form a transmission arena under the indicated channel models:*

$$A_i = \begin{cases} A_i^S & \text{Protocol Model 1,3, Physical Model} \\ A_i^D & \text{Protocol Model 2,3, Physical Model.} \end{cases} \tag{11}$$

The corresponding arena-bounds can be chosen as

$$M = \begin{cases} 1 & \text{for } \Delta > 2, \text{ any Protocol Model,} \\ \lceil \frac{(4+2\Delta)^d}{\Delta^{2d}} - 1 \rceil & \text{for } \Delta \le 2, \text{ Protocol Models 1, 2,} \\ \lceil \frac{(2+\Delta)^d}{\Delta^d} - 1 \rceil & \text{for } \Delta \le 2, \text{ Protocol Model 3,} \\ \left\lceil \frac{3^\alpha P_{\max}}{\beta P_{\min}} \right\rceil & \text{for Physical Model.} \end{cases} \tag{12}$$

Here, $P_{\max}$ and $P_{\min}$ are the maximum and minimum transmission power of the nodes.

[1] More generally we could define an arena-rate function to be any non-negative function ϕ_i such that $\sum_{i \in \mathcal{SD}} \phi_i(X) \le M \cdot W_o$ for any set of transmissions $\mathcal{SD}$ and any $X \in \mathbb{R}^d$.

Note that the above arena-bounds (12) depend only on parameters of the channel model. For the Physical Model one needs to assume that the transmission power $P_k \in [P_{\min}, P_{\max}]$ where $P_{\min} > 0$.

Lemma 2. *Under the Generalized Physical Model, for both A_i^S and A_i^D we may chose the arena-bound*

$$M = \left(\frac{\max_i(W_i)}{B} + \frac{3^\alpha P_{\max} \log_2(2 \cdot \#\mathcal{SD})}{P_{\min}} \right). \tag{13}$$

Note that the arena-bound M of (13) is $O(\max_i(W_i)/B + \log(n))$. If we have $l_i \ge n^\gamma$ for some constant $\gamma < 0$ then $\max_i(W_i) = O(B \log(n))$ and $M = O(\log(n))$. Such γ exists in networks with mesh or random homogeneous topology [15]. Also, If we apply "bounded propagation model" then $\max_i(W_i) = O(B)$ and $M = O(\log(n))$.

5. BOUNDS BASED ON TOPOLOGY

In this section, we compute novel transport capacity bounds which are sensitive to network topology and hold for arbitrary traffic patterns.

While most existing work deals with the geometry of the network only on a very coarse level such as the node density in a homogeneous arrangement, using the concept of transmission arenas we are able to understand and quantify the impact of more detailed spatial information such as clustering of nodes on wireless network capacity.

5.1 Bounds on the Transport Capacity

As a main contribution of this paper and novelty in the field we provide bounds for the transport capacity of an arbitrary wireless network in terms of the length of its Euclidean Minimum Spanning Tree (EMST). An EMST is a tree formed by the network nodes where the weights of the edges are the Euclidean distances of the nodes such that the total weight of the tree is minimum. We establish a useful, simple property first.

Lemma 3. *Consider a continuous curve Γ that connects S_i to D_i. Then, for both, $A_i = A_i^S$ and $A_i = A_i^D$, we have*

$$l_i W_i \le \int_\Gamma \phi_i(X) dX \tag{14}$$

The claim follows easily from the fact that the portion of Γ located inside A_i^S has a length of at least l_i. Similar for A_i^D.

Theorem 1. *Let M be the arena-bound of the underlying channel model. Then, the transport capacity of an arbitrary wireless network is bounded as*

$$C_T \le M \cdot W_o \cdot L_{\text{EMST}} \tag{15}$$

where L_{EMST} is the length of the EMST of the network.

Recall that (12) and (13) give the arena-bound under the three channel models of Section 3. The well-known bounds of [11, 10, 1] are simple special cases of Theorem 1 which applies to a much wider range of networks and to any channel model with an arena-bound such that (14) holds.

Corollary 1. *Let M be the arena-bound of the underlying channel model. Assume that the network nodes are located in a d-dimensional cube with volume V. Then, the transport capacity is bounded as*

$$C_T \le M \cdot W_o \cdot K_d \sqrt[d]{V n^{d-1}} \tag{16}$$

where K_d is a constant that only depends on d.

In order to compute the upper bound in Theorem 1 we do not need to find the EMST of the network. In fact, the length of any curve passing through all nodes can be used instead of L_{EMST}. As a practical application we compute an upper bound for the transport capacity of networks consisting of *clusters* of nodes.

THEOREM 2. *Let M be the arena-bound of the underlying channel model. Assume that the nodes are distributed in n_c clusters which are all of diameter at most l_c. Then, the transport capacity of the network is bounded as*

$$C_T \leq M \cdot W_o \cdot [L_c + K_d l_c \sqrt[d]{n_c n^{d-1}}] \quad (17)$$

where L_c is the length of the EMST over the centers of the clusters. Moreover, if the clusters are located in the cube V,

$$C_T \leq M \cdot W_o \cdot K_d[\sqrt[d]{V n_c^{d-1}} + l_c \sqrt[d]{n_c n^{d-1}}] \quad (18)$$

5.2 Tightness of Spanning Tree Bound

As we establish next, the length of the EMST is indeed a key quantity since the bound of Theorem 1 differs typically from the maximum achievable transport capacity of the network by at most a factor of $O(\log(n))$.

THEOREM 3. *The following holds under all channel models of Section 3 except Protocol Model 3, and under assumption that for the Physical models $\alpha > d$ (see (2) and (3)). Assume a well-connected[2] wireless network with EMST size L_{EMST} is given. Then, there exists a traffic pattern and a time scheduling with transport capacity C_T*

$$C_T \geq K_1 \cdot W_o \cdot L_{\text{EMST}}/K_2 \quad (19)$$

where K_1 is a constant number, also $K_2 = 1$ if $d = 1$ and $K_2 = \log(n)$ if $d \geq 2$.

Note that the bound of Theorem 3 cannot be improved more than a constant if the length of EMST is the only topology information of the network. Figure 2 depicts a network for which $C_T = O(L_{\text{EMST}}/\log(n))$ under any channel model. As for the constant factor, we note that alternative arena functions which capture more minute information about the network topology and channel model could provide tighter bounds.

We complete our discussion of tightness with the following result. Recall that the Maximum Independent Set, denoted by $\text{MIS}(l)$, is a set with maximum size that no two nodes are within distance l from each other.

THEOREM 4. *Assume Protocol Model 3. Then the transport capacity of the network (C_T) is bounded as*

$$C_T \leq \#\text{MIS}(\Delta \cdot \text{r}) W \cdot r \quad (20)$$

Moreover, if the network is connected and large enough so that it is not covered by a single transmission, then there exists a constant $c > 0$, a traffic pattern and a time scheduling with transport capacity C_T

$$C_T \geq c \cdot \#\text{MIS}(\Delta \cdot \text{r}) W \cdot r \quad (21)$$

where $c = \frac{\Delta^d}{2(8+3\Delta)^d}$.

[2]To avoid pathologies we call a network *well-connected*, if there exists constants $0 < \epsilon, \varepsilon < 1$ such that for every arbitrary two nodes in the network there are a path, and a time scheduling for transporting data at rate ϵW_o along the path with transmission power $\varepsilon P_{\max}$.

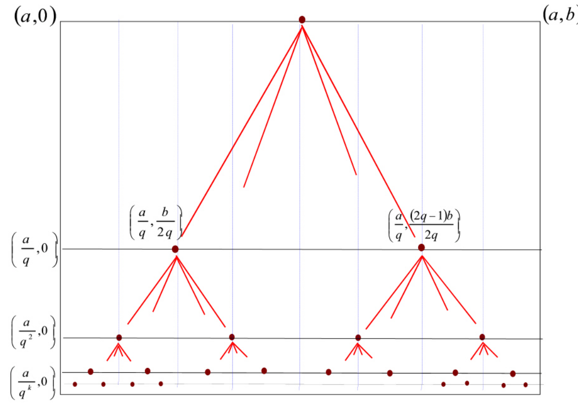

Figure 2: An example for network topology with $C_T = O(L_{\text{EMST}}/\log(n))$. If $q \in \mathbb{N}$ is chosen large enough, a parent node and its children or grand children cannot receive simultaneously successfully from different transmitters. Then, we can show that the transport capacity is bounded by $c \cdot L_{\text{EMST}}/\log_q(n)$ where c is a constant.

6. BOUNDS BASED ON TOPOLOGY AND TRAFFIC PATTERN

In this section, we move from simultaneous direct transmissions to simultaneous multihop flows.

We present a method for computing novel capacity bounds based on both network topology and flow patterns. While the method developed here lends itself to tackle much more general problems, we address two particular cases for illustration purposes: (1) bounding the capacity region of the rates of simultaneous single-path flows (2) bounding the maximum aggregate data rate carried between two parts of the network.

6.1 Bounding Average Rates via Arenas

Let $\mathcal{N}$ denote the set of nodes which are used as senders at certain times along the routes of the a number of flows. For each such sender $S_i \in \mathcal{N}$ let $\mathcal{T}_i$ denote the set of time instants during which it is successfully sending. Finally, let M denote the arena-bound under the channel model put in place. Then, the defining inequality of M (6) implies for any fixed time instant τ:

$$\sum_{S_i \in \mathcal{N}} W_i \mathbb{I}_{A_i}(X) \cdot \mathbb{I}_{\mathcal{T}_i}(\tau) \leq M \cdot W_o \quad (22)$$

Fixing the location X and averaging over the time interval $[0, T]$ we find

$$\sum_{S_i \in \mathcal{N}} < W_i > \mathbb{I}_{A_i}(X) \leq M \cdot W_o, \quad (23)$$

where $< W_i >:= \frac{1}{T}\int_0^T W_i \cdot \mathbb{I}_{\mathcal{T}_i}(\tau)$ is the average rate at which the sender S_i transmits data.

In this section we exploit this bound by considering locations X where we expect the most restrictive conditions in the network, i.e., where a large number of arenas overlap, in order to get effective upper bounds on the capacity region.

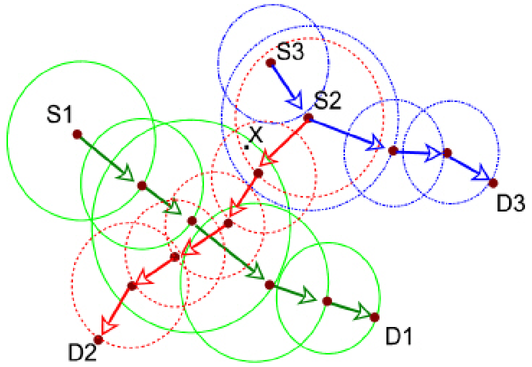

Figure 3: Computing arena-rate functions at particular points provides an upper bound region for the capacity region in a network with simultaneous single path flows.

6.2 Capacity Region of Single-Path Flows

We consider a set of k simultaneous single-path flows and concentrate on bounds obtained via the disturbance areas A_i^S of the transmitters S_i. Lemmas 1 and 2 indicate the applicable channel models as well as the arena-bounds. Similar results will hold when using the protection area of the receivers.

Each location X imposes via (23) a restriction in form of an upper bound on a linear combination of some rates. Thus, the set of all inequalities together provide a convex region in the space of the rates (if there are k flows, the region is a convex set in $\mathbb{R}^k$) which must contain the capacity region of the achievable flow rates in the same space ($\mathbb{R}^k$).

For the sake of concreteness, consider the three single path flows shown in Figure 3. Note that some flows share common nodes like flow 2 and flow 3, or share common space such as at the intersection of flow 1 and flow 2. Now, consider a point X in the intersection of all flows (a point at intersection of two red circles and one green and blue circle) then it gives us this inequality for the $R_1+2R_2+R_3 \leq MW_o$. Here, R_j denotes the flow-rate of the flow j, i.e., $R_j = < W_i >$ for any of the senders S_i along the route of flow j. Note that R_2 appears with coefficient 2 in the inequality, since the point X is located in the disturbance areas of the transmitters of flows 2 twice.

How tight the upper bound is which we obtain using (23) with some fixed location for the capacity region of the rates with this methodology depends on the topology, traffic pattern and even the channel model. Note that without further assumptions the upper bound which is computed by this method can be arbitrarily larger than the actual capacity region of the rates. Indeed, Figure 4 shows a topology and traffic pattern such that $\sum R_i$ is bounded (the size of the circles is a geometric series), however the upper bound computed by (23) can be arbitrarily large for this setting.

6.3 Maximum Flow Rate between Two Parts

Here, we study the maximum data flow rate between two parts in a network in a special case, namely for the flow rate of one side of a rectangular area to the other as shown in Figure 5. Denoting the length and width of rectangle by l_x and l_y we find that the maximum flow rate from left to right side of the rectangle is bounded by $MW_o \cdot \lceil l_y/2l_x \rceil$.

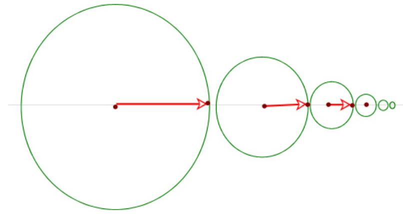

Figure 4: An example of a network topology and a traffic pattern where the upper bounds resulting from (23) are arbitrary larger than the actual capacity region of the rates.

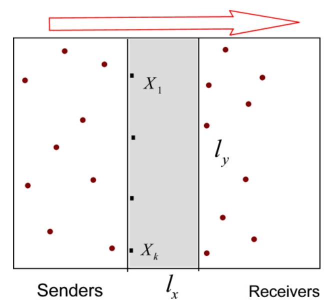

Figure 5: Computing the maximum flow that goes from left to right side of a rectangle area $l_x * l_y$.

For the proof, consider points $X_1, X_2, \ldots, X_k$ close to the left side of rectangle area in distance $2l_x$ from each other. The number of such points is $k = \lceil l_y/2l_x \rceil$. Any transmission from the left to the right side of the rectangle contains at least one of these points X_j in its disturbance area. Summing (23) over all X_k, the sum of transmission rates becomes bounded by $k \cdot MW_o$.

7. MULTICAST ARENAS AND CAPACITY

So far, we dealt with unicast. In this section, we generalize the concept of transmission arena-rate functions and establish novel capacity bounds for multicast applications.

7.1 One-to-Many Arena-rate Function

Consider a transmitter S_i and a set of receivers $D_i^{(1)}, D_i^{(2)}, \ldots, D_i^{(m)}$. In analogy to (8) we define the *one-to-many arena-rate function* of this transmission to be

$$\phi_i(X) = \max_{k=1\ldots m} \phi_i^{(k)}(X) \tag{24}$$

where $\phi_i^{(k)}(X)$ is an arena-rate function of transmission S_i to $D_i^{(k)}$. It is essential but straightforward to note that these multicast arena-rate function possess an arena-bound M in the sense of (6) where M can be chosen as in the unicast case (see Lemmas 1 and 2).

LEMMA 4. *The multicast arena-rate function satisfies (6) with the same arena-bound M as the $\phi_i^{(k)}(X)$.*

For a proof, note that the multicast arena-rate function $\phi_i(X)$ at point X equals the arena-rate function of one of the receivers $D_i^{(k)}, D_i^{(2)}, \ldots, D_i^{(m)}$. This leads us to the same arena-bound.

In order to tackle the capacity of multicast applications we require a quantity analogous to the curve integral (14).

While the curve integral is tailored to the one-dimensional transport of information inherent to unicast applications, this new quantity should be tailored to the spatial distribution of information inherent to multicast transmissions. Thus motivated we consider the space integral

$$\sigma_b^\Omega = \frac{1}{W_o} \sum_{i \in \mathcal{H}_b} \int_\Omega \phi_i(X) dX \tag{25}$$

where $\mathcal{H}_b$ is the set of transmissions (whether one-to-one or one-to-many) which transport a given particular bit b of the multicast application and $\Omega \subset \mathbb{R}^d$ is an arbitrary Borel set.

We may think of σ_b as the volume which is "filled" or "spanned" per second for transporting a particular bit. It depends on the locations of the transmissions (routing paths) and the shape of Ω.

7.2 Network Capacity Bounds for Multicast

Note that we sum over successive hops along a path $\mathcal{H}_b$ in (25), as opposed to a set of simultaneous transmission as in (6). Thus, we need to average over time in order to obtain bounds via (6):

LEMMA 5. *Assume that data bits $\{b_1, b_2, \ldots\}$ have been generated and transported to their destinations in time interval $[0, T]$. Then,*

$$\frac{1}{T} \sum_{b_k} \sigma_{b_k}^\Omega \leq M W_o \cdot \int_\Omega dX \tag{26}$$

Now, we consider an application $\mathtt{app}$ in the network for which we have at our disposal a lower bound $s_{\mathtt{app}}$ on the long term average $\sigma_{b_k}^\Omega$; again, b_k denote the bits transported by the application $\mathtt{app}$.

THEOREM 5. *Assume that $s_{\mathtt{app}} \leq (\sum_{b_k} \sigma_{b_k}^\Omega)/(\sum_{b_k} 1)$ as $T \to \infty$ where $\{b_k\}_k$ are the transported bits under application $\mathtt{app}$. Then the rate of generation of successfully transported bits of the application ($\lambda_{\mathtt{app}}$) is bounded as*

$$\lambda_{\mathtt{app}} \leq M W_o \cdot \int_\Omega dX / s_{\mathtt{app}} \tag{27}$$

A particularly simple case in which Theorem 5 applies is found for multicast applications in which every bit is transported along the same path $\mathcal{H}$. Indeed, one may then set $s_{\mathtt{app}} = \sigma_b^\Omega$ and the bound (27) becomes essentially governed by the ratio of the volume of the entire network space to the weighted volumes of the arenas involved. This leads to:

COROLLARY 2. *Consider a wireless network with n nodes which are distributed as a grid in a square area A. Assume the application $\mathtt{app}$ is to multicast from a random node to n_m randomly selected nodes in the network. Then,*

$$\lambda_{\mathtt{app}} = O(W_o M \sqrt{\frac{n}{n_m}}) \quad a.s. \tag{28}$$

as $n \to \infty$.

Corollary 2 generalizes existing results on unicast and broadcast capacity [11] and [13] which are obtained by setting $n_m = 1$, respectively $n_m = n$.

8. CONCLUSION AND FUTURE WORK

We introduced the novel concept of transmission arenas which allows to study the effect of topology and traffic patterns on the capacity of wireless networks in much more detail than existing work.

The key property behind all results is the existence of an arena-bound which imposes limitations on simultaneous transmissions in a compact, analytically tractable way. The simplicity and effectiveness of our methodology comes from the fact that we take a spatial approach where arena-rate functions indicate the impact at every location in the network space caused by simultaneous transmissions.

The arena-bound imposed at every location and time is used in three ways: Fixing a location and averaging over time we find capacity bounds on simultaneous flows which are more accurate or computationally less demanding than standard methods. Fixing time and averaging along appropriate curves we provide novel bounds on the transport capacity which involve the topology of the network via the Euclidean Minimum Spanning Tree or via clustering information and are therefore more accurate than existing bounds. Finally, averaging over time and space we find bounds on multicast applications which have not existed until now.

Our work applies to the three classical channel models, the Protocol, the Physical and the Generalized Physical Models. Many of our results apply actually to any channel model for which an arena-bound can be established.

Acknowledgment

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Appendix

Proof of Lemma 1: Consider an arbitrary point $X \in \mathbb{R}^d$ at time τ and assume that $\phi_{i_1}(X), \ldots, \phi_{i_m}(X) > 0$. If $m = 1$ the upper bound is trivial. So, assume that $m > 1$ and $l_{i_1} \leq l_{i_2} \leq \ldots \leq l_{i_m}$ (see Figure 6). We find the arena-bound M for the models separately as follows:

Protocol Model (1): First we obtain an inequality through double application of triangular inequality. Consider two transmitters S_{i_j} and S_{i_k}, then

$$\begin{aligned}|S_{i_j} - S_{i_k}| &\geq \frac{1}{2}(|S_{i_j} - D_{i_k}| - |S_{i_k} - D_{i_k}| \\ &+ |S_{i_k} - D_{i_j}| - |S_{i_j} - D_{i_j}|) \\ &\geq \frac{1}{2}((1+\Delta)(l_{i_j} + l_{i_k}) - l_{i_j} - l_{i_k}) = \frac{\Delta}{2}(l_{i_j} + l_{i_k})\end{aligned}$$

The inequality shows that the balls with radiuses $\Delta l_{i_j}/2$ around the transmitters S_{i_j} are disjoint in $\mathbb{R}^d$ for $j = \{1, 2, \ldots, m\}$. Now, we consider a bigger ball with radius $(1 + \Delta/2)l_{i_m}$ around point X, it covers the balls around $S_{i_1}, \ldots, S_{i_m}$. So, we have

$$\sum_{j=1}^{m} \pi_d(\Delta l_{i_j}/2)^d < \pi_d(1+\Delta/2)^d l_{i_m}^d. \quad (29)$$

where π_d is the volume of unit sphere in $\mathbb{R}^d$.

On the other hand, we assumed that D_{i_1} receives successfully, (i.e. $\phi_{i_1}(X) > 0$). From the channel model we have

$$(1+\Delta)l_{i_m} < |S_{i_m} - X| + |X - S_{i_1}| + |S_{i_1} - D_{i_1}| \leq l_{i_m} + 2l_{i_1}.$$

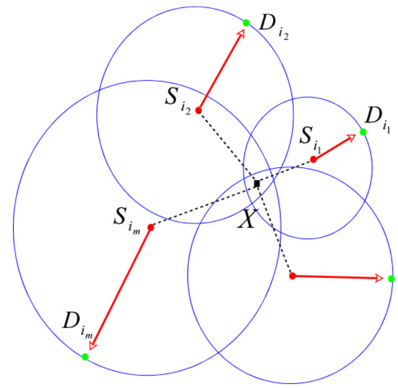

Figure 6: Point X is in the intersection of the disturbance areas of some simultaneous transmission.

From above it follows that

$$l_{i_m} \leq \frac{2}{\Delta} l_{i_1}. \quad (30)$$

Clearly, if $\Delta > 2$ then (30) is a contradiction which shows that $m = 1$ and we set $M = 1$ for this case. If $\Delta \leq 2$, (29) and (30) imply that $m(\Delta l_{i_1}/2)^d < (\frac{2(1+\Delta/2)}{\Delta} l_{i_1})^d$, hence $m < \frac{(4+2\Delta)^d}{\Delta^{2d}}$. Therefore, we set $M = \lceil \frac{(4+2\Delta)^d}{\Delta^{2d}} - 1 \rceil$.

Protocol Model (2): Here, the problem formulation becomes the same as for Model (1), using A_i^D instead of A_i^S. So, we obtain the same M.

Protocol Model (3): We prove a stronger result here. Consider $\psi_i(X) = W_i \cdot \mathbb{I}_{|X - S_i| \leq r}$. Clearly, $\phi_i(X) \leq \psi_i(X)$ when $\phi_i(X)$ is defined in terms of arena A_i^S. Now, we show that $\sum_i \psi_i(X) \leq M \cdot W_o$. Similar to the proof of Model 1, we show if $\Delta > 2$ then $M = 1$. If $\Delta \leq 2$, we consider the balls with radius $\Delta r/2$ around the transmitters S_{i_j}. These balls are disjoint and contained in the bigger ball with radius $(1 + \Delta/2)r$ around point X. It follows that $m < \pi_d(1 + \Delta/2)^d r^d / (\pi_d \Delta r/2)^d = \frac{(2+\Delta)^d}{\Delta^d}$. Therefore, we can set $M = \lceil \frac{(2+\Delta)^d}{\Delta^d} - 1 \rceil$. Note that the same M is obtained if we use this method for arena A_i^D.

Physical Model: Here, we bound the SINR of receiver D_{i_m} as the following

$$|S_{i_k} - D_{i_m}| \leq |S_{i_k} - X| + |X - S_{i_m}| + |S_{i_m} - D_{i_m}| \leq 3l_{i_m}$$

So,

$$\begin{aligned}\text{SINR}_{i_m} &< \frac{P_{i_m} l_{i_m}^{-\alpha}}{\sum_{k=1}^{m-1} P_{i_k} |S_{i_k} - D_{i_m}|^{-\alpha}} \\ &\leq \frac{P_{\max}}{P_{\min}} \frac{l_{i_m}^{-\alpha}}{\sum_{k=1}^{m-1} (3l_{i_m})^{-\alpha}} = \frac{3^\alpha P_{\max}}{(m-1)P_{\min}}\end{aligned}$$

Since, $\text{SINR}_{i_m} \geq \beta$, it follows that $\beta < \frac{3^\alpha P_{\max}}{(m-1)P_{\min}}$, hence $m < \frac{3^\alpha P_{\max}}{\beta P_{\min}} + 1$. Therefore, we can set $M = \lceil \frac{3^\alpha P_{\max}}{\beta P_{\min}} \rceil$.

Proof of Lemma 2: Similar to the proof of Lemma 1, consider $l_{i_1} \leq l_{i_2} \leq \ldots \leq l_{i_m}$. If $m = 1$ the upper bound is trivial, so let us consider the case $m > 1$. We bound the SINR of receiver D_{i_j} for a $1 < j \leq m$ in the following way

$$|S_{i_k} - D_{i_j}| \leq |S_{i_k} - X| + |X - S_{i_j}| + |S_{i_j} - D_{i_j}| \leq 3l_{i_j}$$

for all $k < j$. It follows that

$$\begin{aligned}\mathrm{SINR}_{i_j} &< \frac{P_{i_j} l_{i_j}^{-\alpha}}{\sum_{k=1}^{j-1} P_{i_k}|S_{i_k} - D_{i_j}|^{-\alpha}} \\ &\leq \frac{P_{\max}}{P_{\min}} \frac{l_{i_j}^{-\alpha}}{\sum_{k=1}^{j-1}(3l_{i_j})^{-\alpha}} = \frac{3^\alpha P_{\max}}{(j-1)P_{\min}}\end{aligned}$$

So,

$$W_{i_j} \leq B\log_2(1+\mathrm{SINR}_{i_j}) \leq B\frac{3^\alpha P_{\max}}{(j-1)P_{\min}}\log_2(e) \quad (31)$$

Therefore,

$$\begin{aligned}\sum \phi_i(X) \leq &\; W_{i_1} + \sum_{j=2}^{m} B\frac{3^\alpha P_{\max}}{(j-1)P_{\min}}\log_2(e) \\ \leq &\; \max_i(W_i) + B\frac{3^\alpha P_{\max}}{(j-1)P_{\min}}\log_2(e)\sum_{j=1}^{\#\mathcal{SD}-1}\frac{1}{j} \\ \leq &\; \max_i(W_i) + B\cdot\frac{3^\alpha(1+\log_2(\#\mathcal{SD}))P_{\max}}{P_{\min}}\end{aligned}$$

Proof of Theorem 1: Consider an arbitrary set of transmitter and receiver pairs $\mathcal{SD} = \{(S_1, D_1), (S_2, D_2), \ldots, (S_m, D_m)\}$ at a given time τ. Denote the EMST curve by Γ_{EMST}. This curve connects S_i to D_i for $i = 1, \ldots, m$. From Lemma 3 it follows that

$$\begin{aligned}\sum_{i\in\mathcal{SD}} l_i W_i \leq &\sum_{i\in\mathcal{SD}}\int_{\Gamma_{\mathrm{EMST}}}\phi_i(X)dX = \int_{\Gamma_{\mathrm{EMST}}}\sum_{i\in\mathcal{SD}}\phi_i(X)\;dX \\ \leq &\int_{\Gamma_{\mathrm{EMST}}} MW_o\;dX = MW_o\cdot L_{\mathrm{EMST}}\end{aligned}$$

Proof of Corollary 1: From Theorem 1, it is enough to show that $L_{\mathrm{EMST}} \leq K_d\sqrt[d]{Vn^{d-1}}$. In one dimensional space, the inequality in obvious, because L_{EMST} is the diameter of the network. In two and three dimensional space we use induction for all $n > 1$. We set $K_d = 3d$, and prove the inequality for $n = 2, 3$. Obviously, the maximum distance between two nodes is less than the diameter of the cube $(\sqrt{d}\sqrt[d]{V})$. So, $L_{\mathrm{EMST}} \leq (n-1)\sqrt{d}\sqrt[d]{V} < 3d\sqrt[d]{Vn^{d-1}}$ (only for $n = 2, 3$).

For $n > 3$, we assume the inequality for $k \leq n-1$ and we prove for $k = n$. Consider the balls with radius $1.5\sqrt[d]{\frac{V}{n}}$ around all nodes. At least $\pi_d(1.5\sqrt[d]{\frac{V}{n}})^d/2^d$ volume of each ball is located inside V. The sum of the volumes of these balls inside V is larger than $n\cdot\pi_d(1.5\sqrt[d]{\frac{V}{n}})^d/2^d = \frac{\pi_d 3^d}{4^d}V > V$. Therefore, at least two of these balls are not disjoint that means there exists a pair of nodes within distance $2 * 1.5\sqrt[d]{\frac{V}{n}} = 3\sqrt[d]{\frac{V}{n}}$. We eliminate one of these nodes from the set of the nodes. For the remaining $n-1$, we know the length of the EMST is less than $3d\sqrt[d]{V(n-1)^{d-1}}$. By adding the eliminated node and connecting it to its closet neighbor we build an spanning tree with length of less than $3d\sqrt[d]{V(n-1)^{d-1}} + 3\sqrt[d]{\frac{V}{n}} < 3d\sqrt[d]{Vn^{d-1}}$ (to prove this inequality, show that $d(1-1/n)^{\frac{d-1}{d}} + 1/n < d$ using $0 < 1/n < 1$). Therefore, $L_{\mathrm{EMST}} \leq 3d\sqrt[d]{Vn^{d-1}}$.

Proof of Theorem 2: For a proof of the first part, we construct a spanning tree with the length of less than $L_c + K_d l_c\sqrt[d]{n_c n^{d-1}}$ in two steps. First, we build an EMST over the centers of the clusters. From the assumptions, its length is L_c. Second, we build an EMST over the nodes of each cluster. Denote the number of nodes of the clusters by $m_1, m_2, \ldots, m_{n_c}$. From the assumptions, the diameter of the cluster is less than l_c. This implies that the nodes of each cluster can be placed in a cube with side size l_c. Then, from Corollary 1 we conclude that the length of the EMST which connects the nodes of the k^{th} cluster is less than $K_d\sqrt[d]{m_k^{d-1}l_c^d}$. Hence, the sum the lengths of the EMST's is less than $K_d l_c\sum_{k=1}^{n_c}\sqrt[d]{m_k^{d-1}} \leq K_d l_c\sqrt[d]{n_c n^{d-1}}$ (for a proof of this bound use Jensen's inequality with the concave function $\sqrt[d]{x^{d-1}}$):

$$\frac{1}{n_c}\sum_{k=1}^{n_c}\sqrt[d]{m_k^{d-1}} \leq \sqrt[d]{(\frac{1}{n_c}\sum_{k=1}^{n_c}m_k)^{d-1}} = \sqrt[d]{(\frac{n}{n_c})^{d-1}} \quad (32)$$

The second part is straightforward from Corollary 1.

Proof of Theorem 3: We construct a time scheduling and traffic pattern along some edges of the EMST which achieves the lower bound. We consider $d = 1$ and $d \geq 2$ cases separately.

When $d = 1$, the EMST is the line segment between the two most remote nodes. Due to the well-connectivity assumption, data can be transported between the farthest nodes at rate ϵW_o. This gives us a traffic pattern and time scheduling with a transport capacity larger than $K_1 W_o L_{\mathrm{EMST}}$ where $K_1 = \epsilon$.

When $d \geq 2$, we proceed in five steps.

Step 1: We select the edges of the EMST with the length of at least $L_{\mathrm{EMST}}/2n$. Denote the set of selected edges by $l_1 \leq \ldots \leq l_m$. We have the following inequality for $\sum l_i$:

$$\sum_{i=1}^{m} l_i \geq L_{\mathrm{EMST}} - (n-m)L_{\mathrm{EMST}}/2n > L_{\mathrm{EMST}}/2 \quad (33)$$

Step 2: We partition $\{l_1, ..., l_m\}$ into u sets $\mathcal{C}_1, \ldots, \mathcal{C}_u$ where $\mathcal{C}_j = \{l_i : 2^{-j}L_{\mathrm{EMST}} < l_i \leq 2^{-j+1}L_{\mathrm{EMST}}\}$. Note that $u = \lceil\log_2(2n)\rceil$. We divide the time into u equal time slots. During the j^{th} time slot, the data is transmitted only along the edges of $\mathcal{C}_j$. Next, we do steps 3 to 5 at time slots of $\mathcal{C}_j$ for all $j = 1, 2, \ldots, u$.

Step 3: We divide the space into cube cells with side size $r_j = 2^{-j+1}L_{\mathrm{EMST}}$ such that the coordinates of their centers are $(i_1 r_j, i_2 r_j, \ldots, i_d r_j)$ for $i_1, \ldots, i_d \in \mathbb{Z}$ (see Figure 7).

Such a cellular structure has two properties. (i) If a vertex of an edge $l_i \in \mathcal{C}_j$ is located in a cell, the other vertex of l_i in located either in the same cell or in one of the $3^d - 1$ neighbor cells around it. (ii) The number of edges in set $\mathcal{C}_j$ with at least one of their vertices located in the same cell is bounded by a constant. For a proof, assume that there are k_1 edges of $\mathcal{C}_j$ such that at least one on their vertices is inside the cell. Since the lengths of the edges are less than r_j, the edges are located inside the cube with side size $3r_j$ formed by the cell and the areas its neighbor cells. By Corollary 1, the length of the EMST of the vertices of these edges is less than $K_d\sqrt[d]{(2k_1)^{d-1}3^d r_j^d}$. This is an upper bound for the sum of the lengths of the edges, because if we can connect all the vertices with a spanning tree with smaller length, then we can reduce the length of the EMST of the network which is a contradiction. On the other hand, the length of the edges of $\mathcal{C}_j$ is at least $r_j/2$, so the sum of the length of the edges is larger than $k_1 r_j/2$. We conclude that $k_1 r_j/2 < 3K_d r_j\sqrt[d]{(2k_1)^{d-1}}$, and $k_1 < 2^{2d-1}(3K_d)^d$.

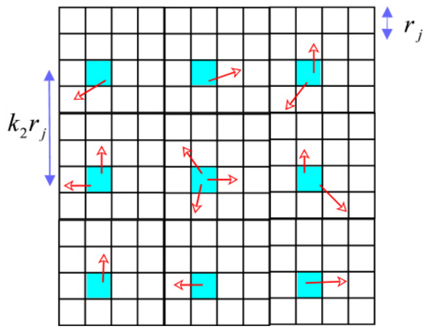

Figure 7: Parameter k_2 is large enough such that the nodes in different cells with same color can transmit simultaneously with rate W_o.

Step 4: We assign color $C(q_{i_1}, q_{i_2}, \dots, q_{i_d})$ to the cell with center $(i_1 r_j, i_2 r_j, \dots, i_d r_j)$, where $q_i = i(\text{mod})k_2$. For time scheduling, we divide the time slot of $\mathcal{C}_j$ into k_2^d subslots and we assign one color to each subslot. At every time subslot only the cells with the corresponding color can be active. A cell is called active when a vertex inside the cell transmits.

The constant k_2 is chosen large enough such that data can be transmitted along some edges $\mathcal{C}_j$ in different cells simultaneously and with rate W_o. We can set k_2 for the different channel models as follows: for the first and second Protocol Models $k_2 = \lceil 3 + \Delta \rceil$, for the Physical Model $k_2 = \lceil 1 + (\frac{\beta}{1-\varepsilon^\alpha} \sum_{J \in \mathbb{Z}_o^d} |J|^{-\alpha})^{1/\alpha} \rceil$ and for the Generalized Physical Model $k_2 = \lceil 1 + (\frac{2^\epsilon - 1}{1-\varepsilon^\alpha} \sum_{J \in \mathbb{Z}_o^d} |J|^{-\alpha})^{1/\alpha} \rceil$ where $\mathbb{Z}_o^d = \{J \in \mathbb{Z}^d : |J| > 0\}$. Note that here we assume that $\alpha > d$ to have $\sum_{J \in \mathbb{Z}_o^d} |J|^{-\alpha} < \infty$. Also, the well-connectivity assumption restricts the length of the edges of the EMST and guarantees that k_2 is finite. For a proof, consider two vertices (nodes) of the edge l_m (the edge maximum size). Any path which connects these two nodes has an edge with size larger than or equal to l_m, otherwise we can reduce the length of the EMST. So, transmissions along distance l_m with rate ϵW_o must be feasible. It follows that $l_i \le l_m \le (\frac{\varepsilon P_{\max}}{\beta N})^{1/\alpha}$ in Physical Model, and $l_i \le l_m \le (\frac{\varepsilon P_{\max}}{(2^\epsilon - 1)N_0})^{1/\alpha}$ in Generalized Physical Model. These inequalities are used for computing k_2.

Step 5: Since, we need at most k_1 transmission along the edges of $\mathcal{C}_j$ for each cell, we divide the subslot of the cell into k_1 equal subsubslots. In each time subsubslot, data is transmitted along one of the edges.

For this traffic pattern and time scheduling scheme the average transmission rate along the edges $\{l_1, ..., l_m\}$ equal to $W/(k_1 k_2^d u)$. This yields a transport capacity of at least:

$$\begin{aligned} C_T &\ge W_o/(k_1 k_2^d u) \sum_{i=1}^{m} l_i > W_o/(k_1 k_2^d u) \cdot L_{\text{EMST}}/2 \\ &\ge K_1 W_o \cdot L_{\text{EMST}}/\log(n) \end{aligned}$$

where K_1 is a constant number.

Proof of Theorem 4: We showed in the proof of Lemma 1 that the distance of the transmitters of two successful simultaneous is larger than $\Delta \cdot r$. So, the maximum number of successful simultaneous transmissions is bounded by $\#\text{MIS}(\Delta \cdot \text{r})$. Therefore, $C_T \le \max \sum W_i l_i \le \max \sum W_i r \le \#\text{MIS}(\Delta \cdot \text{r}) W r$.

For proving the lower bound, first we show that $\#\text{MIS}((4+\Delta) \cdot \text{r}) \ge c\#\text{MIS}(\Delta \cdot \text{r})$ where $c = (\frac{\Delta}{8+3\Delta})^d$. To this end, consider the balls with radius $(4+\Delta) \cdot r$ around the nodes of a $\text{MIS}((4+\Delta) \cdot \text{r})$ of the network. These balls cover all nodes (otherwise, if one node is not covered, then it can be added the MIS which is contradiction). Similar to the proof of Lemma 1, we can show that the number of nodes of a $\text{MIS}(\Delta \cdot \text{r})$ inside each ball is less than $\pi_d((4+\Delta)r + \Delta r/2)^d / (\pi_d(\Delta r/2)^d) = (\frac{8+3\Delta}{\Delta})^d$. Next, we denote the nodes of the $\text{MIS}((4+\Delta) \cdot \text{r})$ by $\{S_1, \dots, S_m\}$. Since the network is connected, for every S_i there exists a node D_i in 2-hop distance such that $r \le |S_i - D_i| < 2r$. Note that data can be transported from S_i to D_i in 2-hop simultaneously and successfully for all $i = 1, \dots, m$. This achieves a transport capacity larger than $\frac{1}{2}\#\text{MIS}((4+\Delta) \cdot \text{r}) W r \ge \frac{c}{2}\#\text{MIS}(\Delta \cdot \text{r}) W r$.

Proof of Theorem 5: Consider the transported bits in the time interval $[0, T - T_d]$ ($T >> T_d$) where T_d is the maximum delay for transporting a bit under application $\mathtt{app}$. From the assumption, the number of the generated bits in this interval is $\lambda_{\mathtt{app}}(T - T_d)$. Then, we have

$$\begin{aligned} (T - T_d)\lambda_{\mathtt{app}} \cdot s_{\mathtt{app}} &= \sum_{k=1}^{\lambda_{\mathtt{app}}(T-T_d)} s_{\mathtt{app}} \le \sum_{k=1}^{\lambda_{\mathtt{app}}(T-T_d)} \sigma_{b_k}^{\Omega} \\ &= \sum_{k=1}^{\lambda_{\mathtt{app}}(T-T_d)} \sum_{i \in \mathcal{H}_{b_k}} \int_{\Omega} A_i(X) dX \\ &\le T \cdot \max_{\mathcal{SD}} \sum_{j \in \mathcal{SD}} \int_{\Omega} \phi_j(X) dX \\ &\le TMW_o \cdot \int_{\Omega} dX \end{aligned}$$

The theorem is proved by letting $T \to \infty$.

Proof of Corollary 2: We set $\Omega = A$. Then, we only need to show that setting $s_{\mathtt{app}} = cA\sqrt{\frac{n_m}{n}}$ satisfies the condition of Theorem 5 almost surely as n grows.

To prove $s_{\mathtt{app}} \le (\sum_{b_k} \sigma_{b_k}^{\Omega})/(\sum_{b_k} 1)$ a.s., we show that $s_{\mathtt{app}} \le \sigma_{b_k}^{\Omega}$ with high probability as n grows. Consider n_m random destinations of a particular data bit b_k. Using probability theory techniques, we can show that there exists a constant c_1 such that $\#\text{MIS}(2\sqrt{\frac{A}{n_m}}) > c_1 n_m$ with high probability as n_m grows (see Lemma 2 in [14]). We draw circles with radius $\sqrt{\frac{A}{n_m}}$ around the nodes of MIS. Clearly, the circles are disjoint in the plane. Next, for every transmission, we color all the grid squares (side size $= \sqrt{\frac{A}{n}}$) which at least $\frac{\pi}{4}$ of their area is located inside the disturbance area of the transmission. It is easy to show that if we connect the neighbor colored squares, then a path from the transmitter to the receiver (receivers in one-to-many transmissions) is created. This shows that there exists a path of colored squares which connects MIS to a node outside of the circle corresponding to it. So, the number of colored squares inside the circle is at least $\lfloor \sqrt{\frac{A}{n_m}} / \sqrt{\frac{A}{n}} \rfloor$. Since the circles are disjoint, we conclude that the number of colored squares is larger than $c_1 n_m * \lfloor \sqrt{\frac{A}{n_m}} / \sqrt{\frac{A}{n}} \rfloor \ge c_2 \sqrt{n n_m}$. This shows that the sum of disturbance areas of the transmissions which transport a particular bit is larger than $c_2 \sqrt{n n_m} \frac{\pi}{4} \frac{A}{n} = cA\sqrt{\frac{n_m}{n}}$.