

ELEC 531: STATISTICAL SIGNAL PROCESSING
Department of Electrical and Computer Engineering
Rice University

Fall 2007

Problem Set 2

Due: September 13, 2007

Problem 2.1 Determine the mean and correlation function of each of the processes X_t defined below.

(a) X_t is defined by the following equally likely sample functions.

$$\begin{aligned} X_t(\omega_1) &= 1 & X_t(\omega_3) &= \sin \pi t \\ X_t(\omega_2) &= -2 & X_t(\omega_4) &= \cos \pi t \end{aligned}$$

(b) X_t is defined by $X_t = \cos(At + \theta)$, where A and θ are statistically independent random variables. θ is uniformly distributed over $[0, 2\pi)$ and A has the density function

$$p_A(A) = \frac{1}{\pi(1 + A^2)}$$

Problem 2.2 (a) Which of the following are valid correlation functions? Indicate your reasoning.

$$\begin{aligned} \text{(i)} \quad R_X(\tau) &= e^{-|\tau|} - e^{-2|\tau|} & \text{(ii)} \quad R_X(\tau) &= \frac{5 \sin 1000\tau}{\tau} \\ \text{(iii)} \quad R_X(\tau) &= \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \leq T \\ 0 & \text{otherwise} \end{cases} & \text{(iv)} \quad R_X(\tau) &= \begin{cases} 1 & |\tau| \leq T \\ 0 & \text{otherwise} \end{cases} \\ \text{(v)} \quad R_X(\tau) &= \delta(\tau) + 25 & \text{(vi)} \quad R_X(\tau) &= \delta(\tau + 1) + \delta(\tau) + \delta(\tau - 1) \end{aligned}$$

(b) Which of the following are valid power density spectra? Indicate your reasoning.

$$\begin{aligned} \text{(i)} \quad S_X(f) &= \frac{\sin \pi f}{\pi f} & \text{(ii)} \quad S_X(f) &= \left(\frac{\sin \pi f}{\pi f} \right)^2 \\ \text{(iii)} \quad S_X(f) &= \exp \left\{ -\frac{(f - f_0)^2}{4} \right\} & \text{(iv)} \quad S_X(f) &= e^{-|f|} - e^{-2|f|} \\ \text{(v)} \quad S_X(f) &= 1 + 0.25e^{-j2\pi f} & \text{(vi)} \quad S_X(f) &= \begin{cases} 1 & |f| \leq 1/T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Problem 2.3 (a) Show that necessary and sufficient conditions for the stochastic process X_t defined by $X_t = \cos(2\pi f_0 t + \theta)$ to be wide-sense stationary is that the characteristic function $\Phi_\theta(j\nu)$ satisfy

$$\Phi_\theta(j1) = 0 = \Phi_\theta(j2).$$

NOTE: f_0 is a constant and is not a random variable.

- (b) Let $X_t = A \cos(2\pi f_0 t) + B \sin(2\pi f_0 t)$, where A and B are random variables and f_0 constant. Find necessary and sufficient conditions for X_t to be wide-sense stationary.
- (c) Consider the process defined by:

$$X_t = A \cos(2\pi f_0 t)$$

where A is a random variable and f_0 is a constant. Under what conditions is X_t wide-sense stationary?

Problem 2.4 It is desired to generate a wide-sense stationary process with correlation function

$$R_X(\tau) = e^{-|\tau|}.$$

Two methods are proposed.

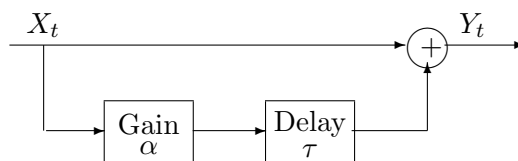
- (i) Let $X_t = A \cos(2\pi Ft + \theta)$ where A , F , and θ are statistically independent random variables.
- (ii) Define X_t by:

$$X_t = \int_0^\infty h(\alpha) N_{t-\alpha} d\alpha$$

where N_t is white and $h(t)$ is the impulse response of the appropriate filter.

- (a) Find two impulse responses $h(t)$ that will work in method 2.
- (b) Specify the densities for A , F , θ in method 1 that yield the desired results. **Hint:** Use results from Problems 1.4 and 2.3.
- (c) Sketch sample functions generated by each method. Interpret your result. What are the technical differences between these processes?

Problem 2.5 Let X_t be a Gaussian random process with mean $m_X(t)$ and covariance function $K_X(t, u)$. The process is passed through the depicted system.



- (a) Is Y_t a Gaussian process? If so, compute the pdf of Y_t .
- (b) What is covariance function of Y_t ?
- (c) If X_t is WSS stationary, is Y_t WSS stationary?
- (d) Compute the cross-correlation function between X_t and Y_t .

Problem 2.6 A process X_t is said to be a *martingale* if it satisfies the relationship

$$E[X_t | X_{u'}, u' \leq u] = X_u, u \leq t.$$

In words, the expected value of a martingale at time t given all values of the process that have been observed up to time u ($u \leq t$) is equal to the most recently observed value of the process (X_u).

- (a) Show that the mean of a martingale is a constant.
- (b) If X_t is a zero-mean martingale having a possibly time-varying variance $\sigma^2(t)$, show that its correlation function is given by

$$R_X(t, u) = \sigma^2(\min(t, u))$$