ELEC 531: STATISTICAL SIGNAL PROCESSING
Department of Electrical and Computer Engineering
Rice University

Fall 2007

Problem Set 5
Due: October 9, 2007

Problem 5.1 Imagine yourself idly standing on the corner in a large city when you note the serial number of a passing beer truck. Because you are idle, you wish to estimate (guess may be more accurate here) how many beer trucks the city has from this single observation.

(a) Making appropriate assumptions, the beer truck's number is drawn from a uniform probability density ranging between zero and some unknown upper limit, find the maximum likelihood estimate of the upper limit.

(b) Show that this estimate is biased.

(c) In one of your extraordinarily idle moments, you observe throughout the city L beer trucks. Assuming them to be independent observations, now what is the maximum likelihood estimate of the total?

(d) Is this estimate of θ biased? asymptotically biased? efficient?

Problem 5.2 *MIMO Channels.* Two parameters θ_1, θ_2 are transmitted over a MIMO (Multiple-Input, Multiple-Output) channel. The two parameters constitute the channel's two-dimensional vector input $\boldsymbol{\theta}$, and the channel output is $\mathbf{H}\boldsymbol{\theta}$. $\mathbf{H}$ is the *non-square* "transfer function" matrix that represents the set of linear combinations of the parameters found in the output. The observations consist of

$$\mathbf{R} = \mathbf{H}\boldsymbol{\theta} + \mathbf{N}\,,$$

where the noise vector $\mathbf{N}$ is Gaussian, having zero mean and covariance matrix K.

(a) What is the maximum likelihood estimator of $\boldsymbol{\theta}$?

(b) Find this estimator's total mean-squared error.

(c) Is this estimator biased? Is it efficient?

Problem 5.3 *Estimating Phase and Amplitude.* You are given the discrete-time signal $A\cos(2\pi f_0 l - \theta)$ observed in the presence of white Gaussian noise having zero-mean and variance σ^2. Eventually, we want to estimate both the phase and the amplitude, but let's build up to that by estimating first the phase θ, then the amplitude A. Throughout assume the number of observations L contains an integer number of periods of the sinusoidal signal and that the frequency f_0 remains fixed.

(a) What is the maximum likelihood estimator for the phase assuming the amplitude and frequency are known?

(b) Find the Cramér-Rao bound for your estimator.

(c) Create a MATLAB simulation of your estimation procedure. Let $A = 1$ and $f_0 = 100/L$, with $L = 1024$. Run 1,000-trial simulations to estimate the phase for $\theta = \pi/4$ for signal-to-noise ratios A^2/σ^2 of 1.0 and 0.04. Calculate the empirical mean and standard deviation of your estimates. Do they agree with theory?

(d) What are the maximum likelihood estimators of the phase and the amplitude? Find the Cramér-Rao bound for this case.

(e) Using the same simulations as in part (c), do these estimates have the predicted statistics?

Problem 5.4 In Poisson problems, the number of events n occurring in the interval $[0, T)$ is governed by the probability distribution

$$\Pr[n] = \frac{(\lambda_0 T)^n}{n!} e^{-\lambda_0 T} ,$$

where λ_0 is the average rate at which events occur.

(a) What is the maximum likelihood estimate of average rate?

(b) Does this estimate satisfy the Cramér-Rao bound?

(c) Now suppose the rate varies sinusoidally according to

$$\lambda(t) = \lambda_0 \exp\{a \cos 2\pi f_0 t\} \quad 0 \le t < T$$

where the frequency f_0 is a harmonic of $1/T$, i.e. $f_0 = K/T$ for some positive integer K. What are the maximum likelihood estimates of λ_0 and a in this case?
Note: The following facts will prove useful.

$I_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{a \cos\theta}\, d\theta$ is the modified Bessel function of the first kind, order 0.

$I_0'(a) = I_1(a)$, the modified Bessel function of the first kind, order 1.

$I_0''(a) = \dfrac{I_2(a) + I_0(a)}{2}$

(d) Find the Cramér-Rao bounds for the mean-squared estimation errors for $\widehat{\lambda}_0$ and $\widehat{a}$ assuming unbiased estimators.