From Signal to Information Processing

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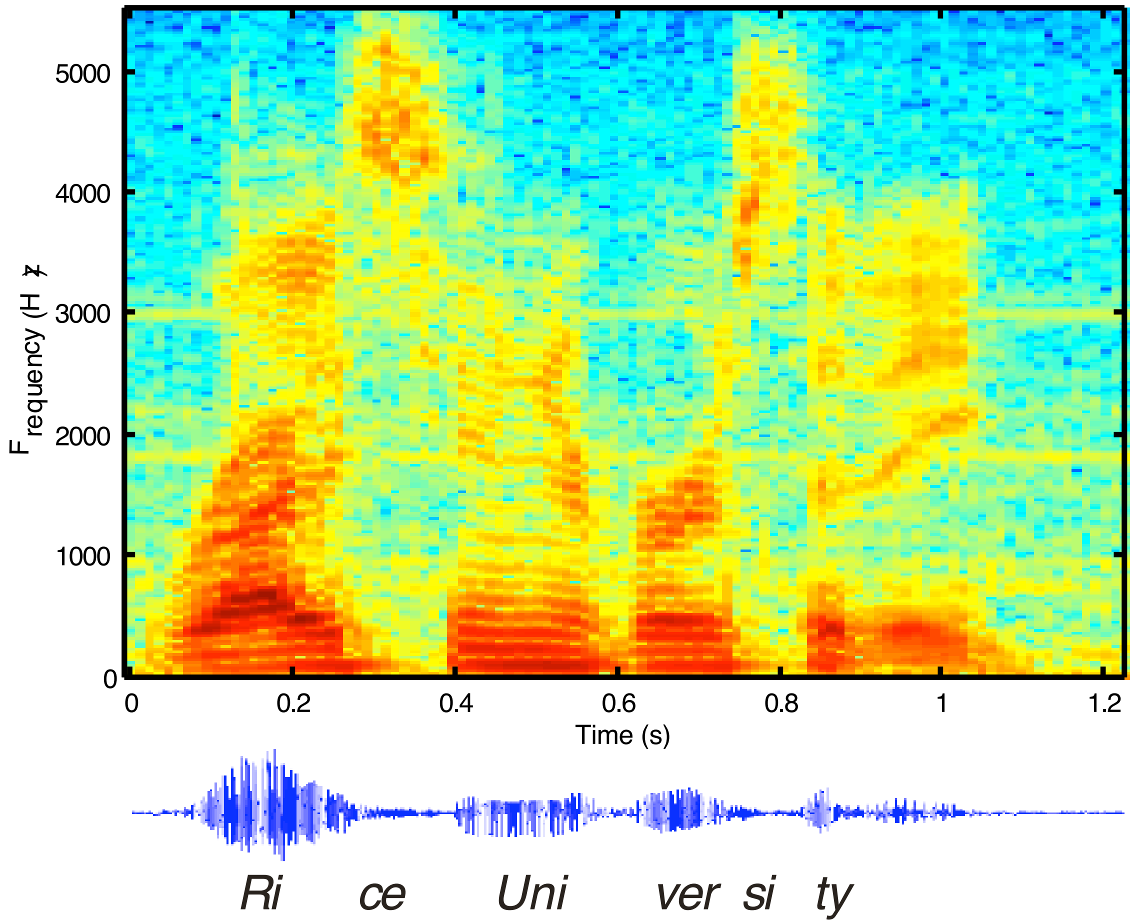
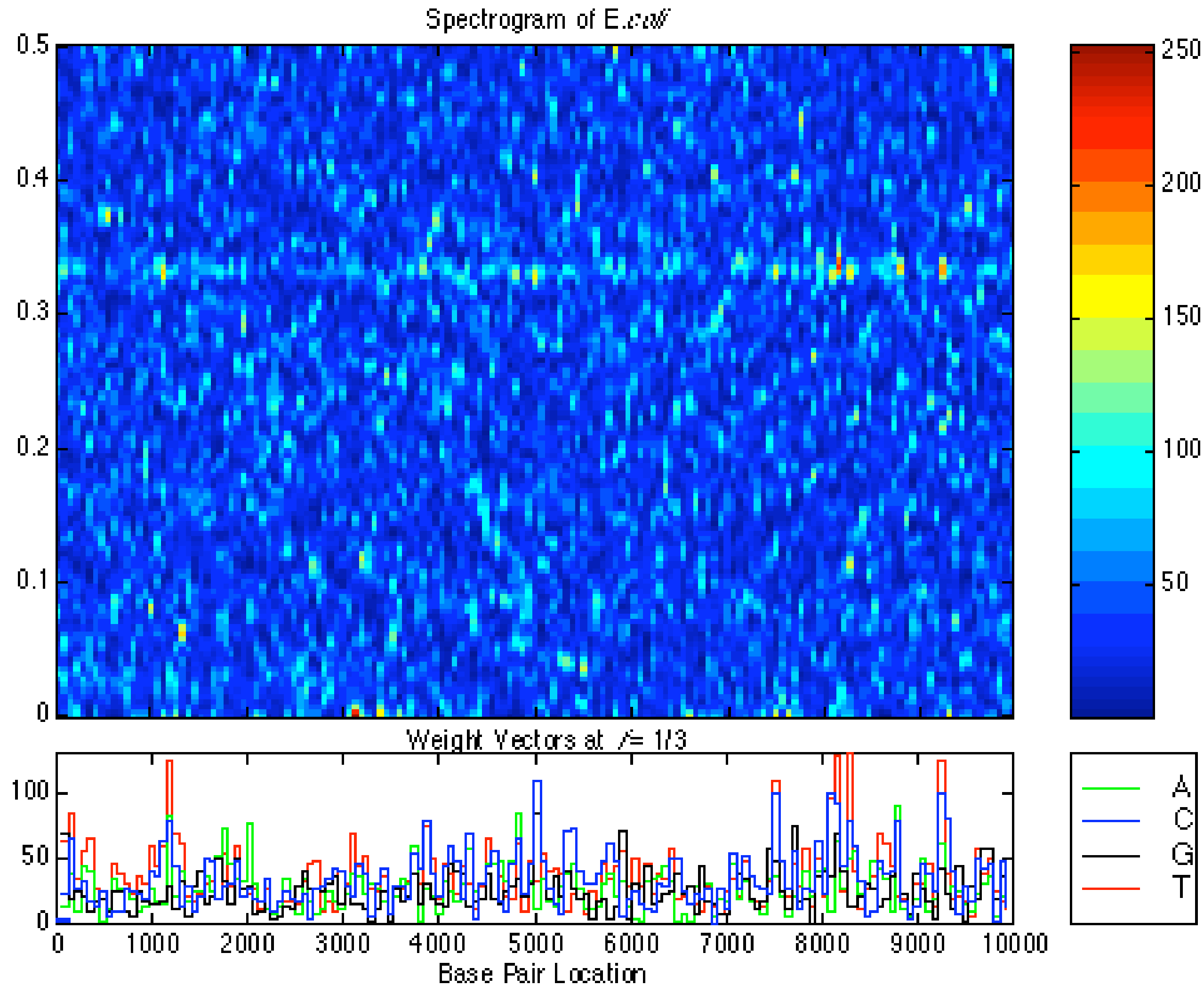
What's the problem?

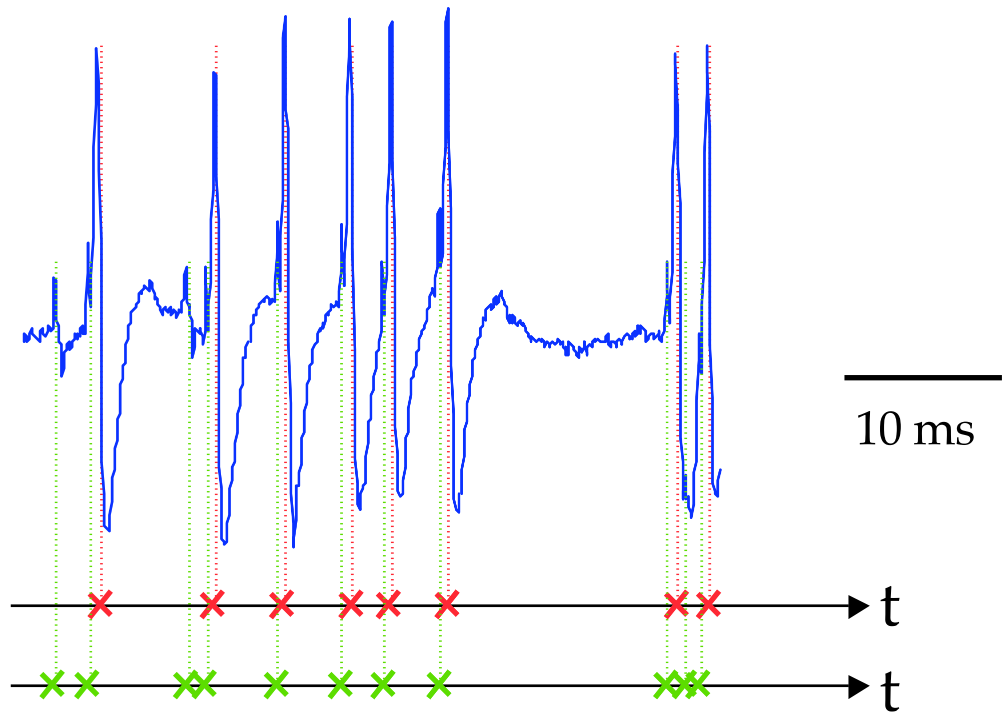
Signal processing has been concerned with form, not
what the signal represents



Not all signals are so easy to analyze

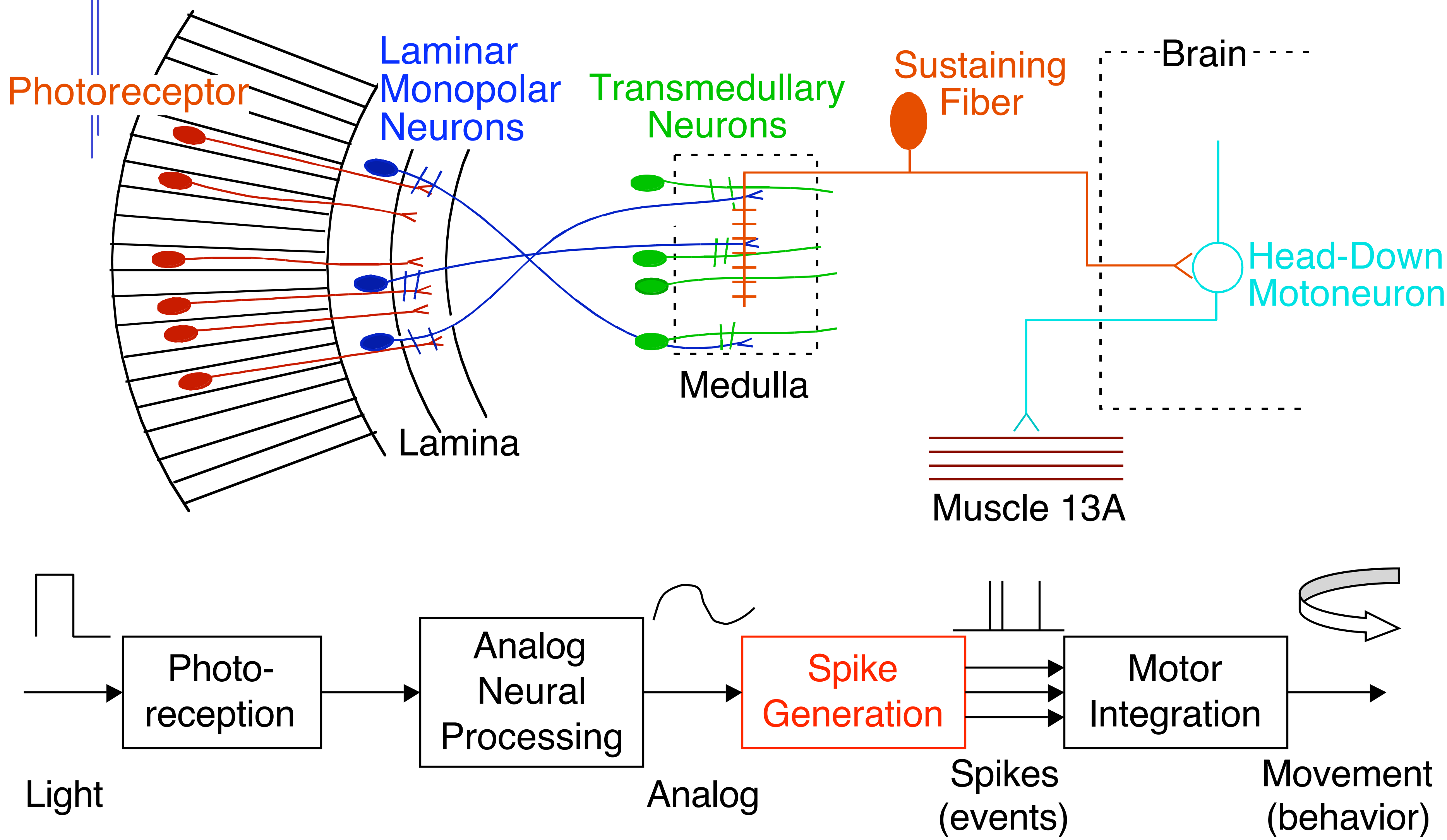


Neural representation of information



Information represented by *when* spikes occur either in single neuron responses

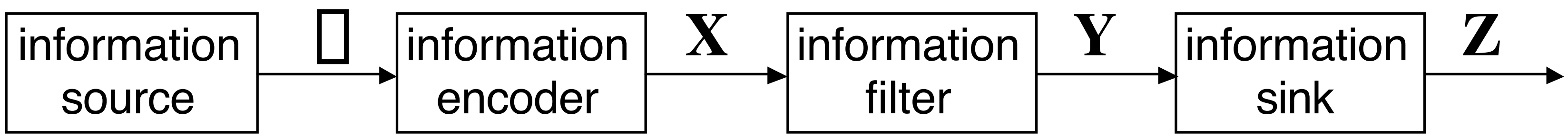
Crayfish dorsal light reflex pathway



Beginnings of information processing

- Information is "in the eye of the beholder"
  - Cellular telephony example (interference to one is information to another)
  - Without interacting with information encoded by a signal, examining signals won't reveal how well (or if) information is represented
- Signals convey information, but how *effectively* to they do so?
- Systems process information, selectively suppressing irrelevant information and accentuating important information by acting on signals (*information filters*)
- System design is usually signal-based, not information based. *What effect does system design have on information processing?*

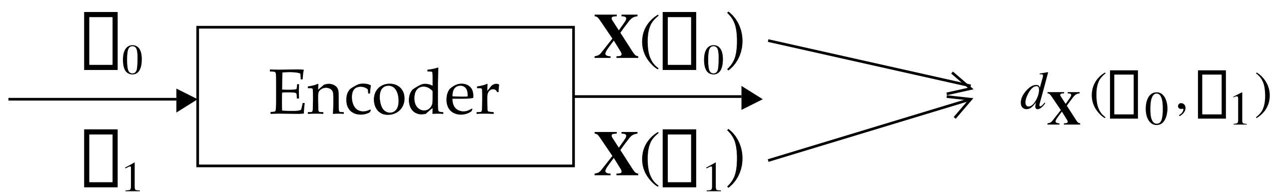
Canonic information processing structure



- Information □ is *always* represented—encoded—by signals
- Systems “process information” indirectly by acting on signals
- Result **Z** is an action or a behavior (i.e., a measurable quantity)
- Any viable information processing theory *must* encompass a variety of signals
- Here, all signals are assumed to be stochastic

Signals represent information

- Let $\square$ represent the information encoded in a signal $\mathbf{X}(\square)$
- Quantify how accurately information changes $\square_0$ $\square$ $\square_1$ are represented by signals with a *distance measure* $d_{\mathbf{X}}(\square_0, \square_1)$



How to choose a distance?

- Calculate distance between the probability distributions $p_{\mathbf{X}}(\mathbf{x}; \square_0)$, $p_{\mathbf{X}}(\mathbf{x}; \square_1)$ characterizing the signal
- Because $p_{\mathbf{X}}(\mathbf{x}; \bullet)$ maps the signal domain to the real-line, we can calculate distances *regardless* of the kind of signal
- Information extraction systems—determining $\square$ from $\mathbf{X}(\square)$—fall into two categories
  - Classification: Which of several values of $\square$ occurred
    Optimal classifier is the likelihood ratio test
    No general formula for performance is known
  - Estimation: Determine $\square$ from a continuum of values
    Mean-squared error a frequently used performance measure

Distances and optimal processing

- The optimal classifier that tries to determine whether θ_0 or θ_1 was encoded will have an error probability of the form $$P_e \sim 2^{-d_{\mathbf{X}}(\theta_0,\theta_1)}$$
- Cramér-Rao lower bound on the mean-square error incurred by *any* (unbiased) estimator

$$E[\varepsilon^2] \geq \frac{1}{F(\theta)} \quad \text{(scalar } \theta\text{)} \qquad E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'] \geq [\mathbf{F}(\boldsymbol{\theta})]^{-1} \quad \text{(vector } \boldsymbol{\theta}\text{)}$$

$$[\mathbf{F}(\boldsymbol{\theta})]_{ij} = E\left[\frac{\partial \ln p_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \ln p_{\mathbf{X}}(\mathbf{x};\boldsymbol{\theta})}{\partial \theta_j}\right]$$

Fisher information matrix

- Fisher information matrix related to distance induced by small information changes (locally Gaussian property)

$$d_{\mathbf{X}}(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0+\delta\boldsymbol{\theta}) \propto K \cdot (\delta\boldsymbol{\theta})^T \mathbf{F}(\boldsymbol{\theta}_0)\,\delta\boldsymbol{\theta}$$

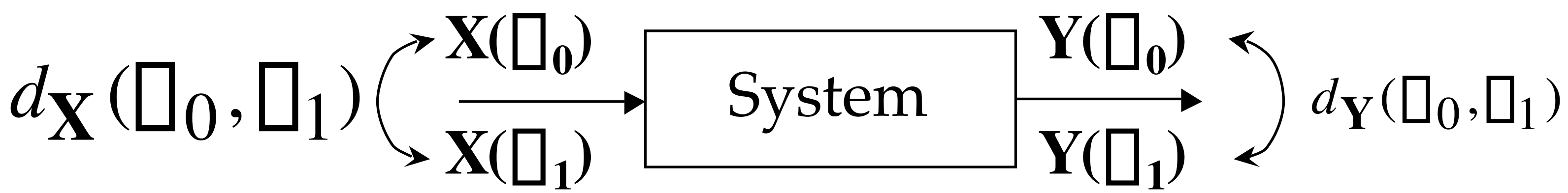
- *With one distance, we can quantify how well information is represented from both classification and estimation viewpoints*

Information processing fundamental

- Information-theoretic distance measures obey the Data Processing Theorem:

$$d_{\mathbf{X}}(\square_0, \square_1) \geq d_{\mathbf{Y}}(\square_0, \square_1)$$

Systems cannot increase how well information is represented by their inputs



Choosing a distance measure

- Many information theoretic distances have the locally Gaussian property
- Only two are known to be related to optimal classifier performance
- We choose distance measures related to the Kullback-Leibler distance

$$D_{\mathbf{X}}(\theta_1 \| \theta_0) = \sum_{\mathbf{x}} p_{\mathbf{X}}(\mathbf{x};\theta_1) \log \frac{p_{\mathbf{X}}(\mathbf{x};\theta_1)}{p_{\mathbf{X}}(\mathbf{x};\theta_0)}$$

- Choose base-2 logarithms, which gives distance “units” of bits.

Properties of K-L distance

- $D_{\mathbf{X}}(\theta_1 \| \theta_0) \geq 0$ Equality only when $p_{\mathbf{X}}(\mathbf{x};\theta_1) = p_{\mathbf{X}}(\mathbf{x};\theta_0)$
- $D_{\mathbf{X}}(\theta_1 \| \theta_0) \neq D_{\mathbf{X}}(\theta_0 \| \theta_1)$ (K - L "distance" is not necessarily symmetric)
- If $\mathbf{X}(\theta)$ has statistically independent components,

$$D_{\mathbf{X}}(\theta_1 \| \theta_0) = \sum_n D_{X_n}(\theta_1 \| \theta_0)$$

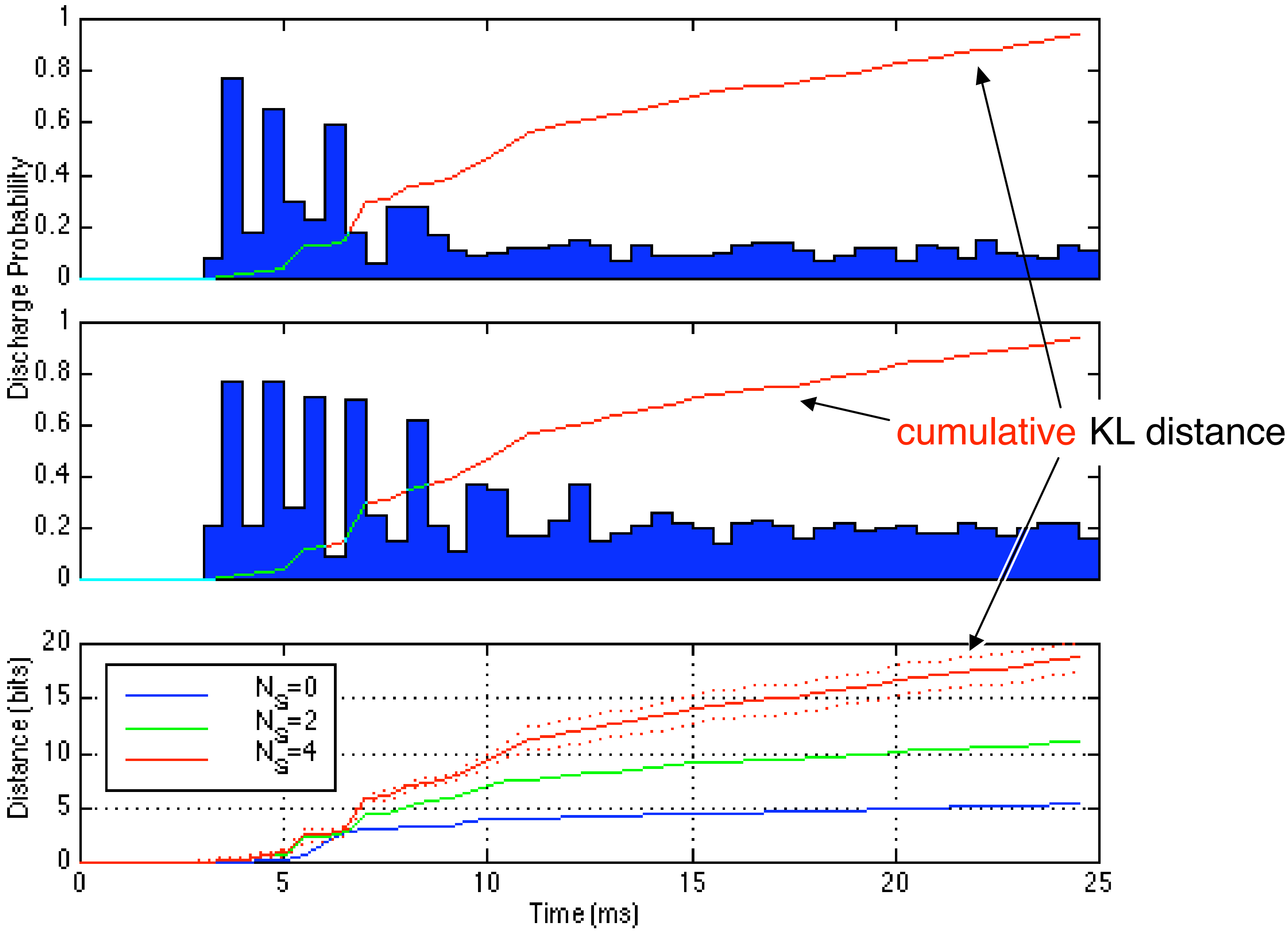
- K-L distance is the "exponential rate" of Neyman-Pearson detector's false-alarm probability

$$P_F \sim 2^{-N D_X(\theta_1 \| \theta_0)} \text{ for fixed } P_M$$

- Distance resulting from information perturbations is "proportional" to Fisher information

$$D_{\mathbf{X}}(\theta_0 + \delta\theta \,\|\, \theta_0) \approx \frac{\delta\theta^{T}\,\mathbf{F}(\theta_0)\,\delta\theta}{2\ln 2}$$

Distance between LSO response patterns



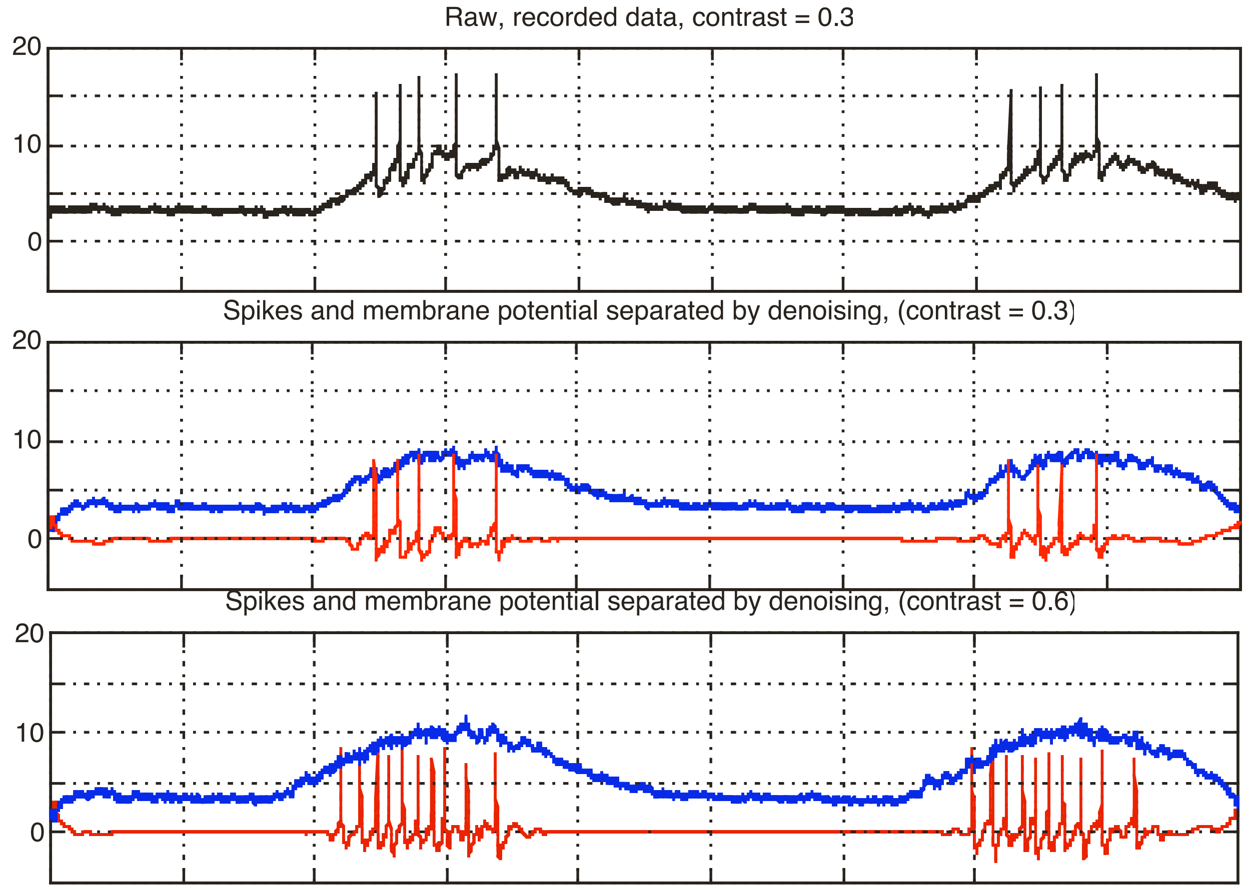
Analyzing system performance

- Quantify a system's information processing performance with the information transfer ratio

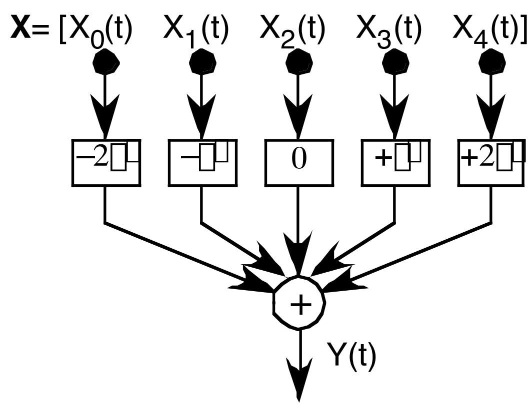
$$\square_{\mathbf{X},\mathbf{Y}}(\square_0,\square_1) = \frac{d_{\mathbf{Y}}(\square_0,\square_1)}{d_{\mathbf{X}}(\square_0,\square_1)}$$

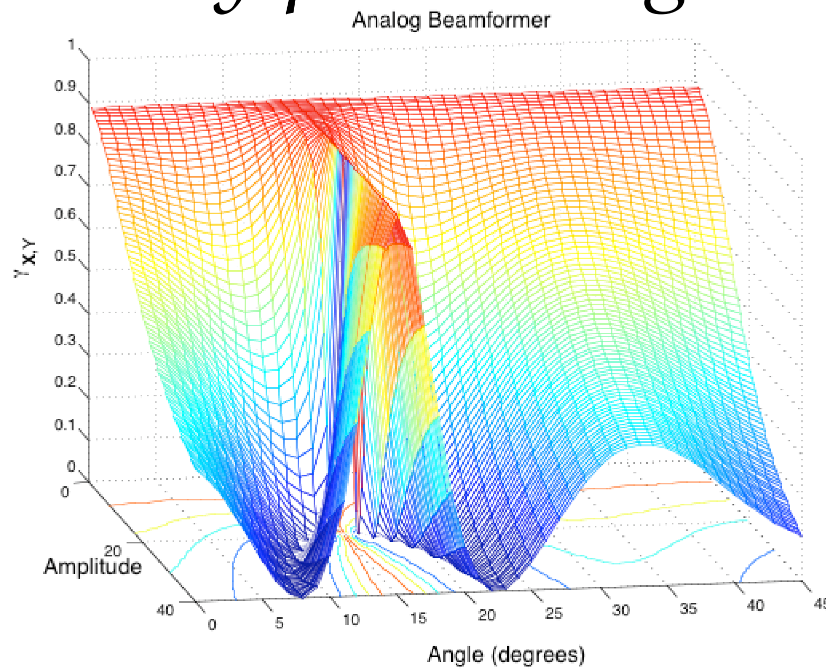
  - $0 \leq \square_{\mathbf{X},\mathbf{Y}}(\square_0 , \square_1) \leq 1$
  - If $\square_{\mathbf{X},\mathbf{Y}}(\square_0 , \square_1) = 1$, the information change is well encoded in the output signal.
  - If $\square_{\mathbf{X},\mathbf{Y}}(\square_0 , \square_1) << 1$, the information change is poorly encoded in the output signal
- Choose a reference $\square_0$; explore how $\square$ varies about this point
- *Information filtering*

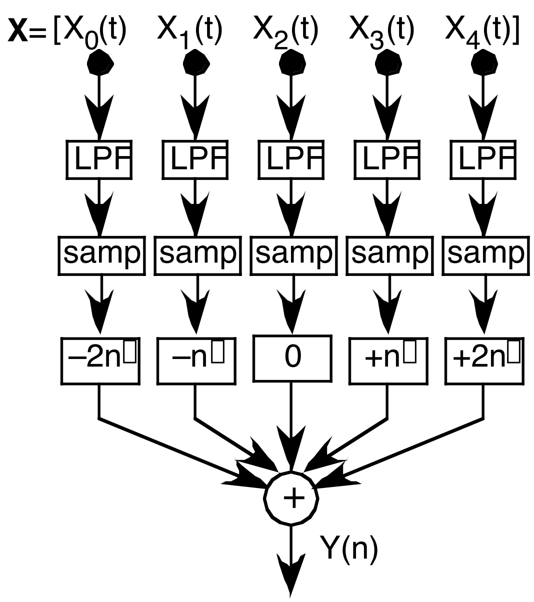
Information transfer across a synapse

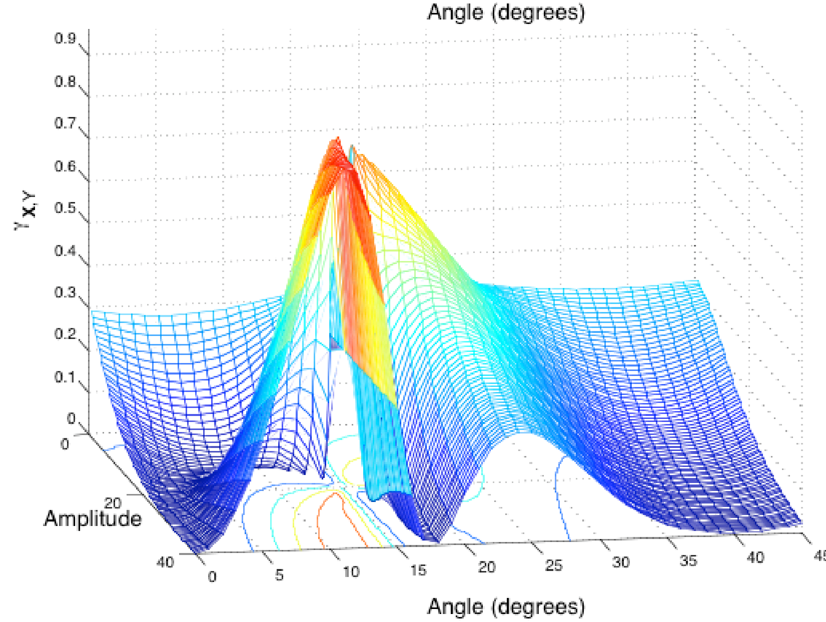


Information filtering: Array processing



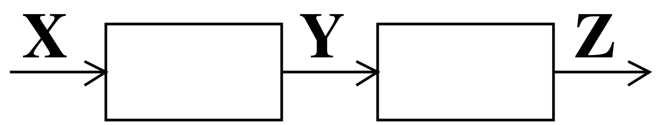






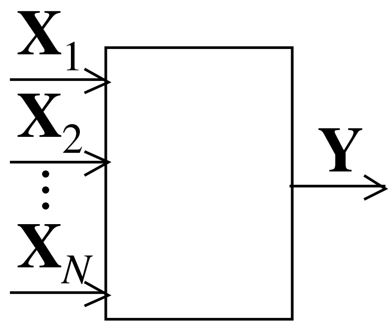
System theory of information processing

Cascade of systems



$$\gamma_{\mathbf{X},\mathbf{Z}} = \gamma_{\mathbf{X},\mathbf{Y}} \cdot \gamma_{\mathbf{Y},\mathbf{Z}}$$

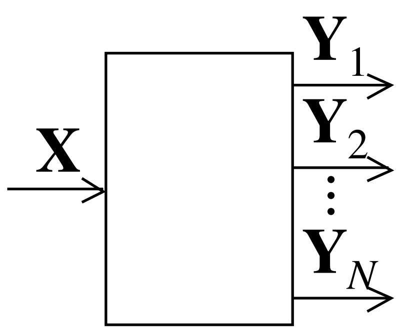
Multiple input systems



If inputs are independent,

$$\frac{1}{\gamma_{\mathbf{X},\mathbf{Y}}} = \sum_n \frac{1}{\gamma_{X_n,\mathbf{Y}}} \Rightarrow \gamma_{\mathbf{X},\mathbf{Y}} \le \min_n \{\gamma_{X_n,\mathbf{Y}}\}$$

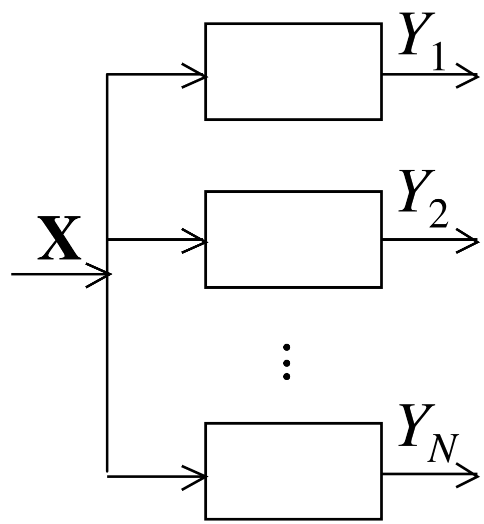
Multiple output systems (*e.g.*, neural populations)



$$\gamma_{\mathbf{X},\{Y_1,\ldots,Y_N\}} = \gamma_{\mathbf{X},Y_1} + \sum_{n=2}^{N} \gamma_{\mathbf{X},\{Y_n | Y_1,\ldots,Y_{n-1}\}}$$

Non-cooperative populations

- The non-cooperative structure defines a baseline for multi-output systems



- The outputs are *conditionally* independent, *not* statistically independent

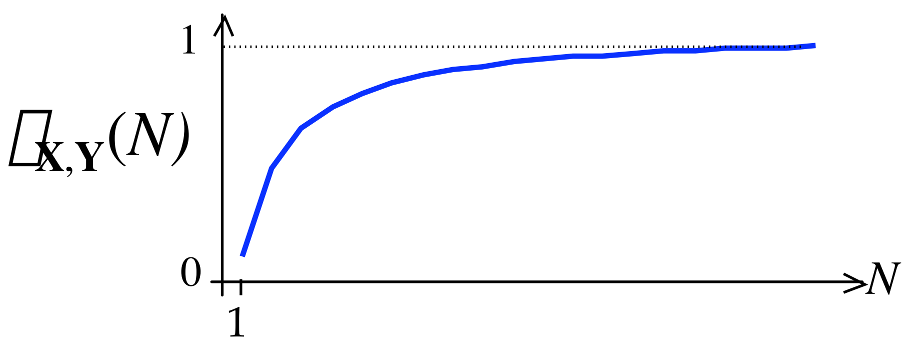
$$p(Y_1, Y_2, \ldots, Y_N; \theta) = \int p(Y_1|\mathbf{x})\, p(Y_2|\mathbf{x}) \cdots p(Y_N|\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x}; \theta)\, d\mathbf{x}$$

- The outputs contain only input-induced dependence

Non-cooperative population theory

- Assume each system is not too noisy ($\square_n \geq \square_{\min} > 0$)
- As the population size N increases, the population can represent the information expressed by its input without loss, regardless of the information representation

$$\lim_{N \square} \square_{\mathbf{X},\mathbf{Y}}(N) = 1$$



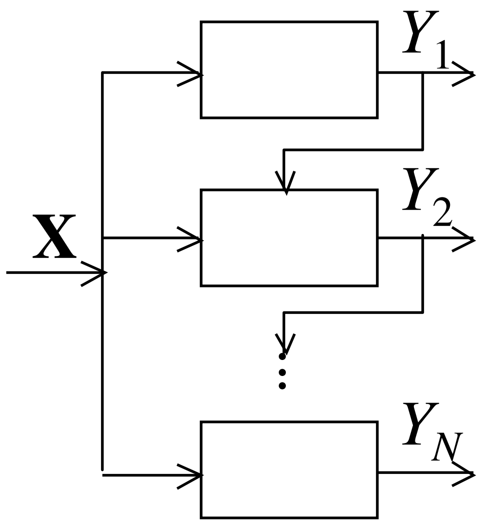
$$\square_{\mathbf{X},\mathbf{Y}}(N) \square 1 \square \frac{k}{N}$$ Continuous code

or

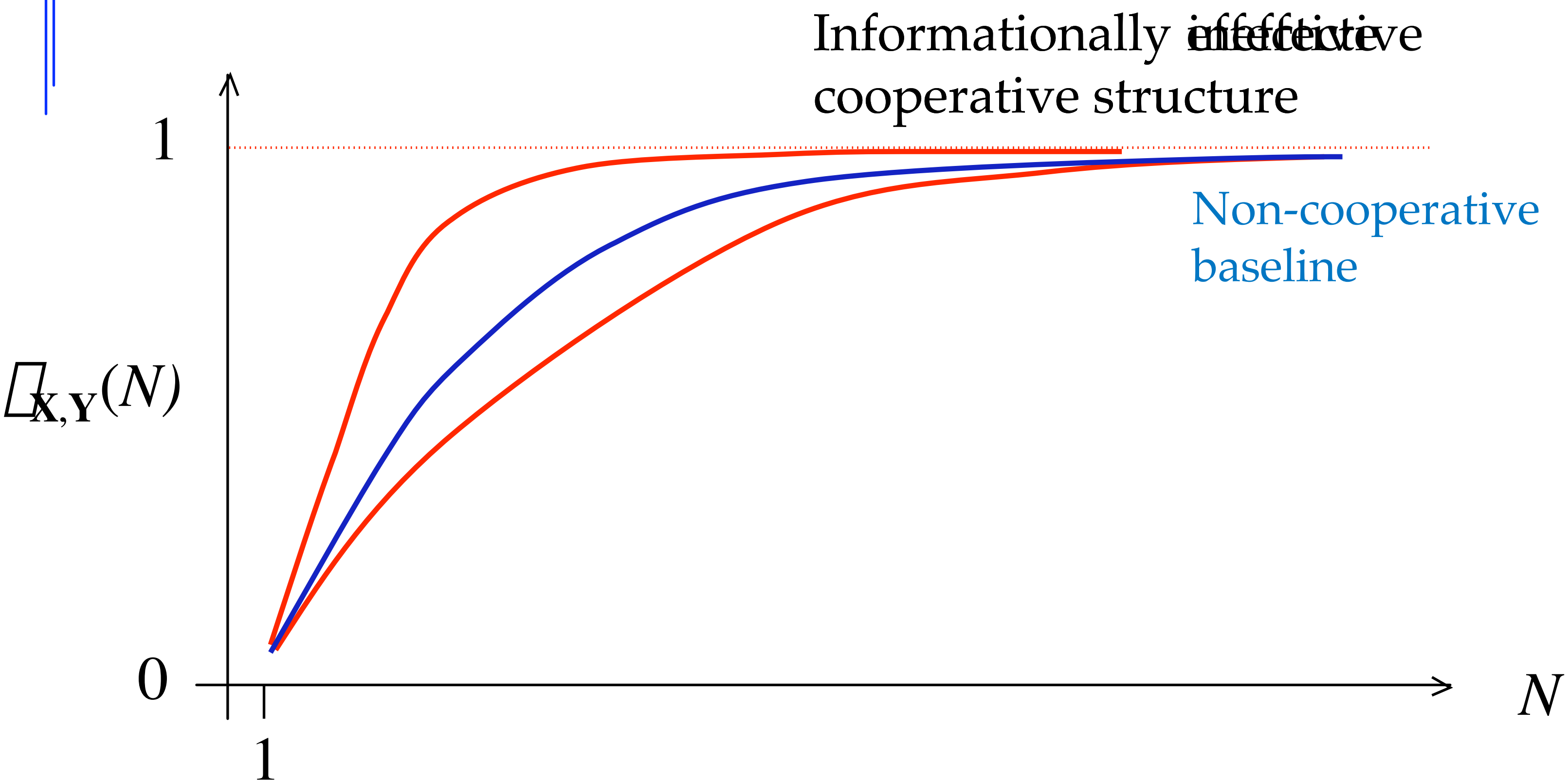
$$\square_{\mathbf{X},\mathbf{Y}}(N) \square 1 \square k_1 e^{\square k_2 N}$$ Discrete code

Cooperative populations

If the cooperation among systems involves output feedback to a limited number of other systems, *the asymptotics of noncooperative systems apply as well*.

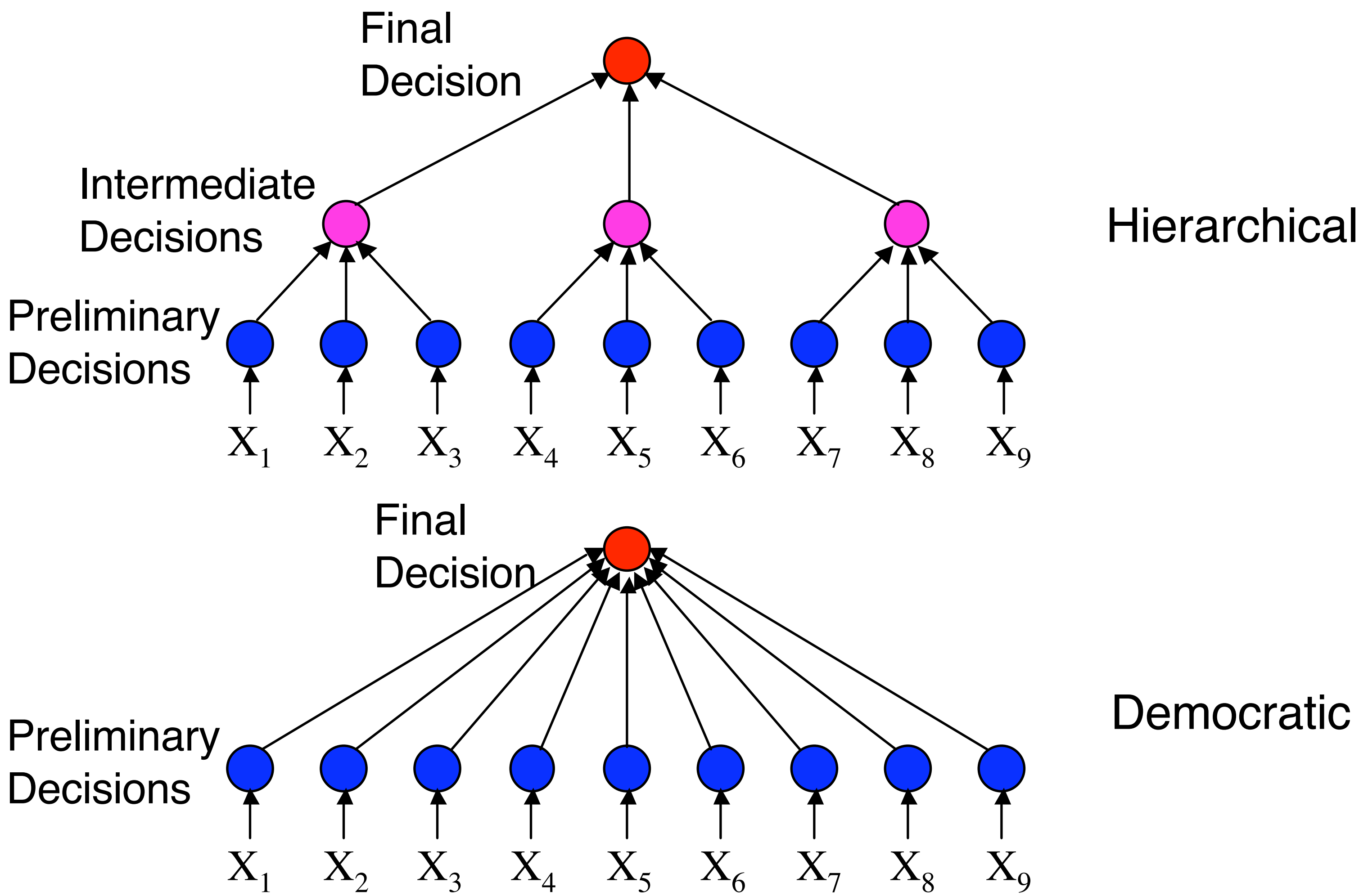


Population coding performance limits

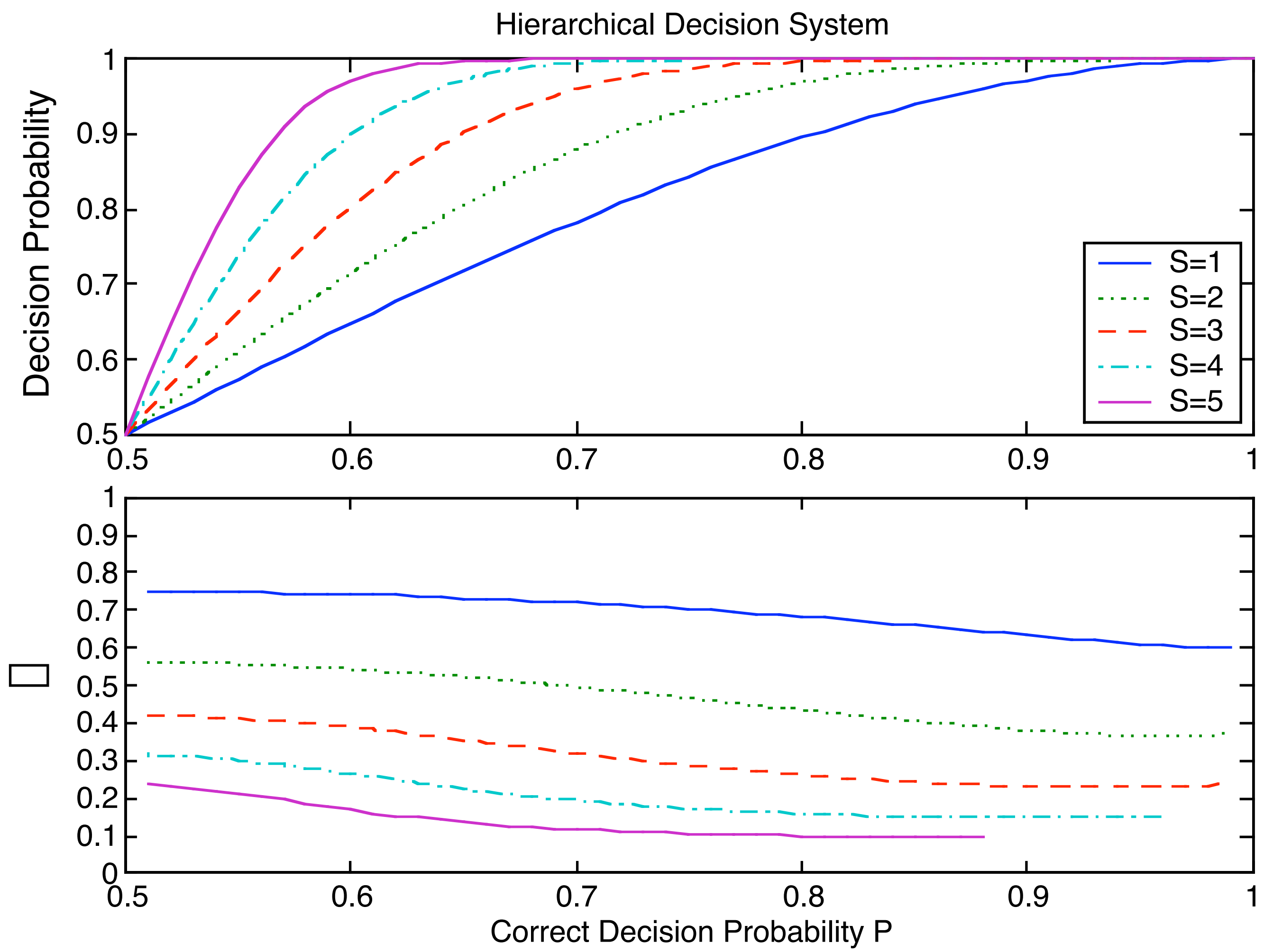


Distributed decision systems

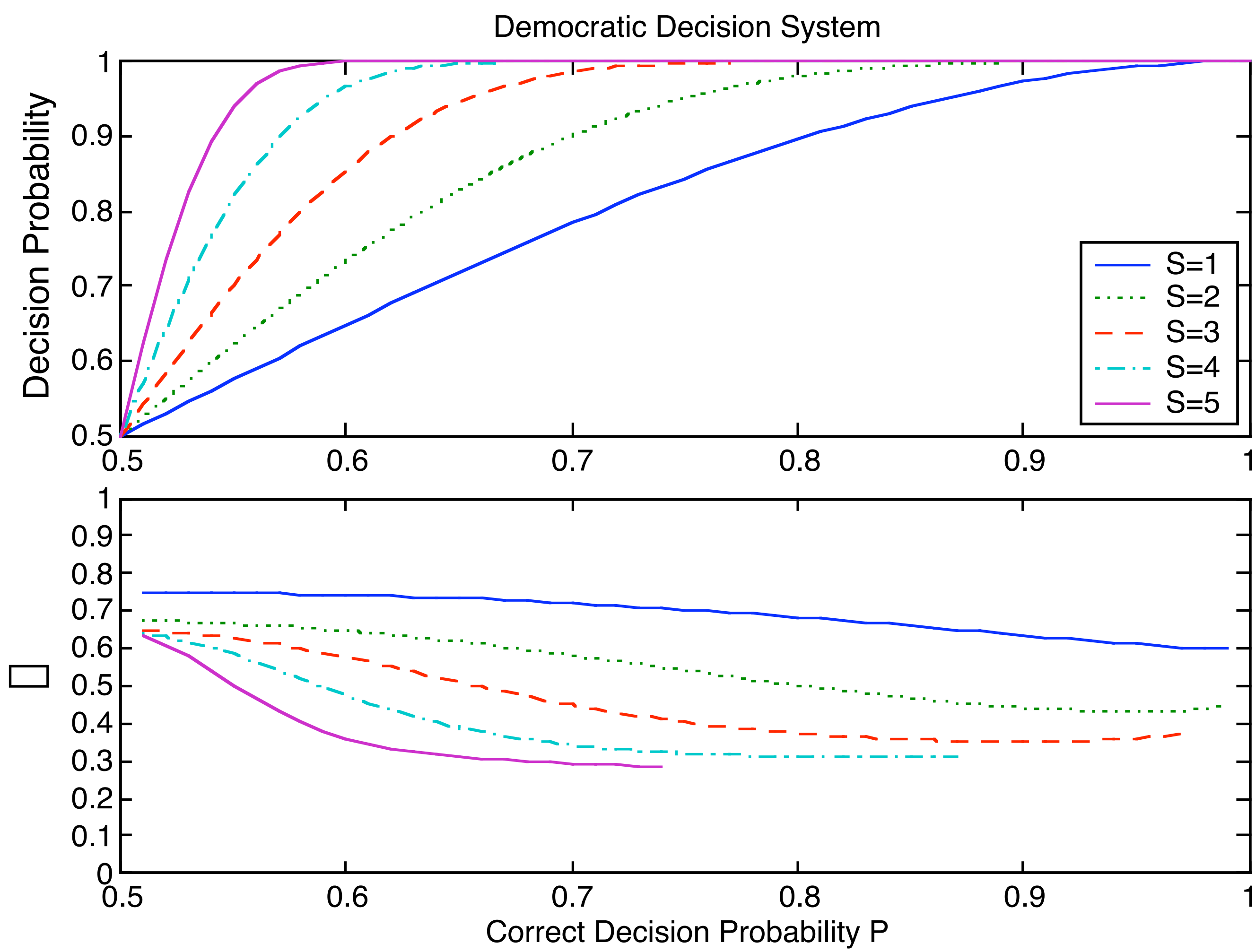
What is the most effective way to integrate individual decisions into a global decision?



Results: Hierarchical structure



Results: Democratic structure



Summary

- A *theory of information processing* must not depend on the nature of the signals representing information
- The theory presented here uses information theoretic distances, particularly the Kullback-Leibler distance, as the primary tool
- Data Processing Theorem is a *fundamental* result that can be widely applied
- Information processing *structures* have fundamental properties regardless of…
  - the information being processed
  - the signals representing the information
- We can assess signal encoding and system processing, hopefully leading to better designs that focus on the *information*, not the signal

Collaborators

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