ELEC 533 Homework 1

Due date: Monday, September 17, 2007

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Total points available: 120
100 points = 100%

1. (5 points) A coin is tossed 12 times. The sequence observed is the 12-tuple (H,H,T,H,H,T,H,H,H,H,T,H). Is this likely a fair coin?

2. (25 points total) A nice way to prove combinatorial identities is to invent a problem whose answer can be computed by counting in two different ways. For instance, the identity $\binom{n}{k} = \binom{n}{n-k}$ can be proved as follows: the left hand side of the equation counts the number of ways of selecting k elements from a set of size n. However, notice that this is exactly the same as computing the number of ways of *rejecting* $n-k$ elements from a set of size n, the quantity denoted by the right hand side.

   Prove the following identities in two ways: a) by ordinary algebraic manipulation and/or mathematical induction; b) by the method of counting in two ways.

   (a) (6 points) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

   (b) (9 points) $(\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

   (c) (10 points) $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$.

3. (25 points total) Given an arbitrary string consisting of m A's, n B's and p C's, compute the probability that an A will be immediately followed by a B

   (a) (10 points) at least k times.

   (b) (15 points) exactly k times. (Hint: think recursively.)

4. (25 points total, 5 points for each part) Verify that the following are valid probability laws. To this end determine whether the underlying probability space is discrete or continuous and use the criteria established in class.

   (a) The binomial law $(0 \le \theta \le 1)$ on $\Omega = \{0, 1, ..., n\}$

$$P[\{a\}] = \frac{n!}{(n-a)!\; a!}\; \theta^a\; (1-\theta)^{n-a}. \tag{1}$$

   (b) The Poisson law $(\lambda > 0)$ on $\Omega = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$$P[\{a\}] = \frac{e^{-\lambda}}{a!}\; \lambda^a. \tag{2}$$

   (c) The exponential law $(\lambda > 0)$ on $\Omega = \mathbb{R}$

$$P[(-\infty, y]] = \begin{cases} \int_0^y \lambda\, e^{-\lambda\tau} d\tau & \text{if } y \ge 0 \\ 0 & \text{if } y < 0. \end{cases} \tag{3}$$

(d) The uniform law on $[a, b]$

$$P[[x, y]] = \frac{y - x}{b - a} \qquad b \geq y \geq x \geq a. \tag{4}$$

(e) The following law on $\mathbb{N} = \{1, 2, \ldots\}$

$$P[\{n\}] = 2^{-n}. \tag{5}$$

5. (10 points total, 5 points each part) Let $\Omega_4 = \{1, 2, 3, 4\}$.

   (a) Find the smallest σ-algebra containing $A = \{1, 3\}$.

   (b) Find the smallest σ-algebra containing $A = \{1\}$, $B = \{3\}$.

6. (10 points) Carrie owns n different pairs of shoes. She is cleaning out her closet and wants to get rid of some shoes in order to make room for new ones. Suppose she chooses $2r$ shoes at random to keep ($r < n/2$). What is the probability that she has not kept a single matching pair?

7. (20 points total) In statistical physics, the state of a system of particles can be approximated by the distribution of the particles over a grid of cells of the phase space. (The state of each particle in the phase space is uniquely characterized by a p-dimensional coordinate, usually $p = 6$, 3 space and 3 momentum coordinates.) Assuming a finite number of states for the observation of the system, the phase space can be represented with a finite number of N (p-dimensional) grid cells.

   For *photons*, the statistical properties of the system is described by the so-called Bose-Einstein statistics. In that model, two states of the system are considered indistinguishable whenever the number of photons in each grid cell is the same for both states, regardless which individual particles are in a given grid cell. It is also assumed that all *distinguishable* states are equally likely. (Note: it is clear that different distinguishable states can be realized by different numbers of indistinguishable states.)

   (a) (5 points) How many distinguishable states can a system of n photons exhibit on a grid of N cells of the phase space? (Note that the number of all possible configurations including the indistinguishable ones is N^n, which is not the number we are asking here.)

   (b) (7 points) What is the probability of a specific grid cell being occupied by exactly $k < n$ photons?

   (c) (8 points) What is the probability of $m < n$ photons collectively occupying M specific cells, $M < N$ (i.e., m photons are distributed in M specific cells)?