ELEC 533 Homework 2

Due date: In class on Wednesday, September 26
85 points total (80 points =100%)
Instructor: Dr. Erzsébet Merényi

8. (20 points total) A sample space is given by $\Omega = \{(x, y) : 0 \le x \le 1, \quad 0 \le y \le 1\}$ Let A, B, C, D, E, F and G be the following events:

$$A = \{(x, y) : y \le 2x \le 1, 0 \le x \le 1/2 \quad \textit{and} \quad y \le 2-2x \le 1, 1/2 \le x \le 1\}$$

$$B = \{(x, y) : 1/2 \le x \le 1, \quad 0 \le y \le 1\}$$

$$C = \{(x, y) : y \le x\}$$

$$D = \{(x, y) : y \le 1 - x\}$$

$$E = \{(x, y) : x \le 1/4\}$$

$$F = \{(x, y) : 1/2 \le y\}$$

$$G = \{(x, y) : 0 \le x \le 1/2, x \le y \le 1/2 \quad \textit{and} \quad 1/2 \le x \le 1, x \le y \le 1\}$$

   (a) (4 points) Compute $P[A|B]$. Are A and B independent? How are these two results related? (Assume that the probabilities of these events are proportional to the areas with which we defined them. Using drawings may be helpful.)

   (b) (3 points) Compute $P[C|D]$. Are C and D independent?

   (c) (3 points) Are C and E independent?

   (d) (5 points) Are the events A, B, and E independent?

   (e) (5 points) Are the events B, F and G independent?

9. (20 points total) Compute the distribution functions —i.e., $F(a) = P[(-\infty, a])$ for *all* $a \in \mathbb{R}$ — for the following distributions. For (9b) verify also, that this is indeed a valid probability law. In all but the last one, the laws are given through a density.

   (a) (5 points) $\mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

$$f(x) = \begin{cases} \frac{1}{2\pi} & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

   (b) (5 points) Cauchy:

$$f(x) = \frac{1}{\pi(1 + x^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

   (c) (5 points) $\exp(\lambda)$: One sided exponential with parameter $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

(d) (5 points) Given is a probility space $\Omega = \{\texttt{head}, \texttt{tail}\}$ with $P[\{\texttt{head}\}] = 1/3$. Given is also a random variable defined as $X(\texttt{head}) = 3$, $X(\texttt{tail}) = -6$. What is the probability law (distribution) on the real line induced by the random variable X? More precisely, compute $P_{\mathbb{R}}[(-\infty, a]] = P_{\Omega}[X \leq a]$ for all a.

10. (10 points total) Due to an internet configuration error packets sent from BigCity to HugeCity are routed through ProblemTown, with probability 3/4. The conditional probability that a packet is dropped given that it is routed through ProblemTown, is 1/3. The conditional probability that a packet is dropped given that it is not routed through ProblemTown, is 1/4.

    (a) (5 points) What is the probability that a packet is dropped?

    (b) (5 points) What is the conditional probability that a packet was routed through ProblemTown given that it was not dropped?

11. (20 points total) Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. Compute the distribution of Y in the following two different ways:

    (a) (10 points) Use $P[Y \leq y] = P[\tan(X) \leq y]$ to show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$. Then infer the density $f_Y(y)$ of the distribution of Y as the derivative of $F_Y(y)$ and conclude that Y is a Cauchy r.v.

    (b) (10 points) The following formula connecting the densities of the distributions of the random variables X and $Y = g(X)$ is valid if g is differentiable and strictly increasing:

$$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

    Use it to derive $f_Y(y)$ for $g(X) = \tan(X)$.

    This problem provides an intuitive interpretation of the Cauchy law: A person is blindfolded, standing in front of a wall. The person spins around a few times. The angle under which the person now "faces" the wall is completely arbitrary, in other words, uniformly distributed. The person starts walking straight. The location where the person will collide with the wall is random, with Cauchy distribution.

12. (15 points total) Given is a r.v. X which is uniformly distributed on $[0, \pi]$. We are interested in the r.v. $Y = g(X)$ where $g(t) = \sin(t)$.

    (a) (5 points) Compute $\mathbb{E}[\sin(X)] = \int \sin(x) f_X(x) \mathrm{d}x$.

    (b) (5 points) Compute the pdf (density) f_Y of Y. (You might find it convenient to compute first the CDF F_Y of Y.)

    (c) (5 points) Compute $\mathbb{E}[Y] = \int y f_Y(y) \mathrm{d}y$. Check whether you got the same answer as in (a).