ELEC 533 Homework 3

Due date: Wednesday, October 3, 2007
70 points total (70 points = 100%)
Instructor: Dr. Erzsébet Merényi

12. (10 points total) Compute expectation and variance of the following random variables:

    (a) (6 points) $X \simeq \mathcal{N}(\mu, \sigma^2)$: Gaussian or normal distribution with parameters $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, which is given through the density

    $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

    (You don't have to show that this is indeed a probability density.)

    (b) (4 points) $X \simeq \exp(\lambda)$: One sided exponential with parameter $\lambda > 0$, which is given through the density

    $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

13. (25 points total) Suppose that X (signal) is a binary r.v. with $P[X = 1] = \alpha$ and $P[X = -1] = \beta = 1 - \alpha$. Suppose the r.v. N is normal, i.e. $N \sim \mathcal{N}(0, \sigma^2)$ (noise). Assume also that N and X are independent, meaning that the events $\{X = a\}$ and $\{N \leq b\}$ are independent for all a and b. We are interested in the r.v. $Y = X + N$ (noisy observation).

    (a) (5 points) Express $P[Y \leq y | X = 1]$ and $P[Y \leq y | X = -1]$ in terms of Gaussian integrals. (Note that $\int_{-\infty}^{t} \exp(-x^2)dx$ has no closed form.)

    (b) (5 points) Using this and the law of total probability find $F_Y(y)$, again in terms of integrals.

    (c) (5 points) Derive a closed form expression for f_Y from F_Y. This is a mixture density; identify it's components in terms of known density functions.

    (d) (5 points) To infer the signal X from the observation Y the following strategy is used:

    - If $Y \geq \gamma$, we infer that $X = 1$,
    - if $Y < \gamma$, we infer that $X = -1$,

    where γ is some number we will decide on later (next question). We make an error in the inference if $Y \geq \gamma$ and $X = -1$, or if $Y < \gamma$ and $X = 1$. Compute the probability p_e that this happens.

    (e) (5 points) Find the γ that minimizes the probability of error p_e.

14. (10 points total)
Let X be a continuous r.v. with *pdf*

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } 0 < a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Let us define the r.v. Y as $Y = X^2$.

(a) (3 points) Compute the expectation $E[Y]$, using the *pdf* of X, $f_X(x)$, given above.

(b) (7 points) Find the *pdf* of Y, then compute $E[Y]$ using now $f_Y(y)$. Is the r.v. Y uniformly distributed, like the r.v. X?

15. (10 points) Let X and Y be independent, uniform r.v.'s with $f_X(x) = \frac{1}{2}$, $|x| < 1$ and zero otherwise and $f_Y(y) = \frac{1}{4}$, $|y| < 2$ and zero otherwise. Compute the *pdf* of $Z = 2X - Y$.

16. (15 points) Assume that X and Y are independent Gaussian random variables with zero mean and variance 1. Compute the distribution of the random variable $Z = \exp(-(X^2 + Y^2)/2)$.
Hint: the transformation from Cartesian to polar coordinates goes as $x = r\sin(\phi)$, $y = r\cos(\phi)$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\phi$.