ELEC 533 Homework 4

Due: October 17, 2007
Total points: 60 (60 points = 100%)
Instructor: Dr. Erzsébet Merényi

17. (15 points) Let (X, Y) be jointly uniform over the unit disc, i.e.

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \le 1, \\ 0 & \text{else.} \end{cases}$$

    (a) (7 points) Are X and Y independent? Show your reasoning.

    (b) (8 points) Let $Z = X^2 + Y^2$. Compute its CDF F_Z. Hint: no integration needed. But if you choose to integrate recall the polar differential $dxdy = rdrd\phi$.

18. (15 points total) Let the joint density of the pair of random variables (X, Y) be given by

$$f_{XY}(x,y) = \begin{cases} y\exp(-xy) & \text{if } x > 1 \text{ and } y > 0 \\ 0 & \text{else.} \end{cases}$$

    (a) (6 points) Compute the marginal densities f_X and f_Y.

    (b) (4 points) Are X and Y independent? Show your reasoning.

    (c) (5 points) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

19. (5 points) Show that if X and Y are uncorrelated random variables then the variance of their sum is the sum of their variances, i.e., that $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$.

20. (10 points total) Two pairs of discrete random variables (U, V) and (X, Y) are given via their joint distributions:

$$P[U=u, V=v] = \begin{cases} 1/2 & \text{if } u=1,\ v=0 \\ 1/6 & \text{if } u=1,\ v=1 \\ 1/12 & \text{if } u=-1,\ v=1 \\ 1/4 & \text{if } u=-1,\ v=0 \end{cases}$$

    and

$$P[X=x, Y=y] = \begin{cases} 7/12 & \text{if } x=1,\ y=0 \\ 1/12 & \text{if } x=1,\ y=1 \\ 1/6 & \text{if } x=-1,\ y=1 \\ 1/6 & \text{if } x=-1,\ y=0 \end{cases}$$

    (a) (5 points) Show that the marginals are the same, that is $F_X = F_U$ and $F_Y = F_V$.

    (b) (5 points) Which pair is independent?

21. (15 points total) Let X and Y be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

    where ρ is a constant.

    (a) (7 points) Show by direct computation that $\mathrm{cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = \rho$.

    (b) (8 points) Compute the density f_Z of the sum $Z = X + Y$ using the general formula for the density of the sum of (dependent) random variables: $f_Z(z) = \int f_{XY}(x, z-x)dx$. Conclude that Z is as well Gaussian.