

ELEC 533 Homework 5

Due date: October 24, 2007

Total points: 60

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23. (20 points total) This problem is the continuation of problem 22 (last problem) from HW4. You already solved the first two subproblems in HW4, those are only repeated for completeness, do not solve them again (hence the 0 points for those subproblems). You may also reuse quantities or expressions that you already computed in HW4 (some means, for example), without calculating again. Just state that fact.

Let X and Y be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

where ρ is a constant.

- (a) (0 points) Show by direct computation that $\text{cov}(X, Y)/\sqrt{\text{Var}(X)\text{Var}(Y)} = \rho$.
- (b) (0 points) Compute the density f_Z of the sum $Z = X + Y$ using the general formula for the density of the sum of (dependent) random variables: $f_Z(z) = \int f_{XY}(x, z-x)dx$. Conclude that Z is as well Gaussian.
- (c) (4 points) Compute the conditional density $f_{Y|X=a}$ using $f_{Y|X=a}(y) = f_{XY}(a, y)/f_X(a)$.
- (d) (4 points) Compute the conditional expectation $\mathbb{E}[Y|X]$ using above density.
- (e) (5 points) Let (U, V) be joint continuous random variables with joint density f_{UV} . Let $W = U+V$. Derive the following formula for the joint density of U and W :

$$f_{UW}(a, c) = f_{UV}(a, c-a). \quad (1)$$

Hint: Express first the joint CDF $F_{UW}(a, b) = P[U \leq a, W \leq b]$ as an appropriate integral using f_{UV} . Then, take derivatives.

- (f) (7 points) Using formula (1) show that U and W are jointly Gaussian, provided U and V are jointly Gaussian. For simplicity assume that U and V have mean zero and variance 1.
24. (12 points) The random variable Y represents the number of file requests arriving at a server in a given time. Let X denote the popularity of files in the following sense: if the popularity of all files was equal, say $X = a$ for all files, then Y would be Poisson with mean a . In reality, the popularity X of files varies on a server and has to be taken as a random variable. Still, conditioned on $X = a$ the number of arrivals will be Poisson, i.e., $Y|X = a$ is Poisson with mean a . To summarize the above the following is all you really need to know to solve this problem: The conditional distribution of Y given X is given by

$$P[Y = k|X = a] = \frac{e^{-a} a^k}{k!}$$

For simplicity¹ we assume here that X is a uniform random variable as in

$$f_X(x) = \begin{cases} 1/5 & 0 \leq x \leq 5 \\ 0 & \text{else} \end{cases}$$

Obviously, Y is not Poisson. Find the unconditional distribution $P[Y = k]$ using the Law of Total Probability

$$P[Y = k] = \int_{-\infty}^{\infty} P[Y = k|X = x]f_X(x)dx$$

¹A more realistic assumption would be a Zipf law for X , i.e., a power law decay for the density of X .

- *Hint* : Use integration by parts to establish a recursive formula which allows to compute $P[Y = k]$ from $P[Y = k - 1]$. Compute $P[Y = 0]$ explicitly. Putting the two together gives an explicit expression for $P[Y = k]$.

25. (18 points total) Consider the experiment of tossing three coins independently. The outcomes are then $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega = \{0, 1\}^3$, where ω_n is the outcome of the n -th toss, $\omega_n = 1$ for heads, $\omega_n = 0$ for tails. Let $X_1(\omega) = \omega_1$, $X_2(\omega) = \omega_2$, $X_3(\omega) = \omega_3$, $Y = X_1 + X_2$ and $Z = X_1 + X_2 + X_3$.

- (a) (2 points) Compute $\mathbb{E}[Z|Y]$.
- (b) (3 points) Using the conditional probability law $P[Z = n|X_1 = x_1 \text{ and } X_2 = x_2]$ we define the conditional expectation in the usual (intuitive) way as

$$\mathbb{E}[Z|X_1 = x_1, X_2 = x_2] := \sum_n n \cdot P[Z = n|X_1 = x_1, X_2 = x_2].$$

Compute this expression as a function $h(x_1, x_2)$. We set $\mathbb{E}[Z|X_1, X_2] = h(X_1, X_2)$.

- (c) (4 points) Show that $\mathbb{E}[Z|Y] = \mathbb{E}[Z|X_1, X_2]$. This means that Y gives the same information towards predicting Z as X_1 and X_2 .
- (d) (3 points) Compute $\mathbb{E}[Z|X_1]$.
- (e) (3 points) Consider the random variable $U = \mathbb{E}[Z|X_1, X_2]$. Compute $\mathbb{E}[U|X_1]$. Compare with 25d.
- (f) (3 points) Compute the variance of the random variables $Z - \mathbb{E}[Z|X_1]$ and $Z - \mathbb{E}[Z|X_1, X_2]$ and compare them.

This illustrates that the variance of error grows as we take a guess at Z with less knowledge.

26. (10 points) Let $X \simeq \mathcal{N}(\mu, \sigma^2)$ be a Gaussian r.v. and set $Y = e^X$. Y is called log-normal.

- (a) (4 points) Compute $\mathbb{E}[Y]$ in terms of μ and σ .
- (b) (2 points) Compute $\mathbb{E}[Y^q]$ for any $q \in \mathbb{R}$. Hint: Write $Y^q = (e^X)^q = e^{qX}$ and use (a).
- (c) (2 points) Use this to compute $\text{Var}(Y)$.
- (d) (2 points) Jensen's inequality says that $\mathbb{E}[\exp(X)] > \exp(\mathbb{E}[X])$. Check this fact by explicit comparison of $\mathbb{E}[Y]$ and $\mathbb{E}[X]$.