ELEC 533 Homework 6

Due date: October 31, 2007
109 points total (90 points = 100%)
Instructor: Dr. Erzsébet Merényi

27. (15 points total) A binary communication channel transmits the signal described by the r.v. X, $X = \{x1 = 0, x2 = 1\}$, with $p(x1) = p1$, $p(x2) = p2$, $p1 + p2 = 1$ (i.e., X takes two values, 0 and 1, with probabilities $p1$ and $p2$, respectively). The signal is corrupted by additive noise N: we receive $Y = X + N$ at the output. The noise N is uniformly distributed over [-1,1]. Assuming $p1$=$p2$=0.5,

    (a) (3 points) Compute the conditional density of $f_{Y|X}(y|x)$, for $X = 0$ and for $X = 1$.

    (b) (3 points) Compute the density of Y, $f_Y(y)$.

    (c) (3 points) Compute the probability that X was sent given that $Y = y$ was observed, i.e., compute $P[X = x_k|Y = y]$ for $k = 1, 2$.

    (d) (3 points) Compute the non-linear MMSE estimator of Y given X.

    (e) (3 points) Compute the non-linear MMSE estimator of X given Y.

28. (16 points total)

    (a) (3 points) Let $X \simeq \exp(\lambda)$ be a one sided exponential random variable with parameter $\lambda > 0$, which is given through the density

    $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

    Compute the characteristic function Φ_X of X.

    (b) (5 points) Let $X_i \simeq \exp(\lambda_i)$ be independent, one sided exponential random variables for $i = 1, \ldots, n$, the density of X_i given by

    $$f_{X_i}(x_i) = \begin{cases} \lambda_i e^{-\lambda_i x_i} & \text{for } x_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

    Is the sum $\sum_{i=1}^{n} X_i$ of n independent exponential random variables an exponential random variable?

    (c) (8 points) Let X and Y be independent exponential r.v. with different parameters: $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$. Show that the minimum of X and Y has an exponential distribution, i.e. $M(\omega) = \min(X(\omega), Y(\omega)) \sim \exp(\xi)$. What is the value of ξ? (Hint, start with $P[M > b]$.)

29. (16 points total)

    (a) (5 points) Let X be Cauchy r.v. which is given through the density

    $$f_X(x) = \frac{1}{\pi(1 + (x - \mu)^2)} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

    Verify that the characteristic function $\Phi_X(u)$ is given by

    $$\Phi_X(u) = e^{iu\mu - |u|}.$$

    Hint: Think indirect verification instead of direct computing of $\Phi_X(u)$.

    (b) (5 points) Let $Y \simeq \mathcal{C}(a, b)$[1] be a general Cauchy r.v., meaning that $Y = a + bX$, where X is the same as in part (a) and where a and b are constants. Compute the characteristic function Φ_Y of Y. (Pay attention to the sign of b, i.e. $\Phi_X(u) = \Phi_{-X}(u) = \Phi_X(-u)$ by symmetry.)

[1] This notation is not common and should only be used in this problem set.

(c) (3 points) Let $Y_1 \simeq \mathcal{C}(a_1, b_1)$ and $Y_2 \simeq \mathcal{C}(a_2, b_2)$ be independent Cauchy r.v.'s. Is the sum $Y := Y_1 + Y_2$ also Cauchy, i.e., is it of the form $Y \simeq \mathcal{C}(a, b)$. Compute a and b from a_1, a_2, b_1 and b_2.

(d) (3 points) The definition of a (symmetrical) α-stable r.v. $(0 < \alpha \leq 2)$ is: $X \simeq S\alpha S(\mu, \sigma)$ if

$$\Phi_X(u) = e^{(i\mu u - |\sigma|^{\alpha}|u|^{\alpha})},$$

where μ is the position parameter, and σ is the scale parameter. (Note that a Cauchy r.v. is $S\alpha S$ with $\alpha = 1$.) Let $X \simeq S\alpha S(\mu, \sigma)$ and assume that $\alpha > 1$. Show that $\mu = \mathbb{E}[X]$. (Note: When $\alpha \leq 1$, then $\mathbb{E}[|X|] = \infty$. Hence $\mathbb{E}[X]$ is not defined.)

30. (18 points total) The members of the sequence of jointly independent random variables $X[n]$ have *pdf*s of the form

$$f_{X_n}(x) = (1 - \frac{1}{n})\frac{1}{\sqrt{2\pi}\sigma}exp[-\frac{1}{2\sigma^2}(x - \frac{n-1}{n}\sigma)^2] \quad + \frac{1}{n}\sigma exp(-\sigma x)u(x)$$

where $u(x)$ is the unit step function.

(a) (8 points) Does the random sequence $X[n]$ converge in the mean-square sense?

(b) (6 points) Does the random sequence $X[n]$ converge in probability?

(c) (4 points) Does the random sequence $X[n]$ converge in distribution? If so, what is the limiting distribution?

31. (12 points total) Let X_n be Gaussian r.v.-s with mean μ_n and variance σ_n^2.

(a) 4 points) Under what conditions on the sequences μ_n and σ_n^2 does X_n converge in distribution and what is the limiting distribution. Hint: Use the fact that X_n converges in distribution if and only if their characteristic functions ϕ_{X_n} converge. Check, under what conditions the limit of ϕ_{X_n} exists and make sure that the limit is again a meaningful characteristic function.

(b) (4 points) Let now X_n have means and variances as $\mu_n = \mu$ and $\sigma_n^2 = \sigma^2/(1 + 2n)$. This sequence of X_n then converges in distribution according to a). Does it converge in probability?

(c) (4 points) Does the sequence X_n converge in the mean square sense?

32. (12 points total) With fixed λ, for each integer $n \geq \lambda$, let $X_{1,n}$, $X_{2,n}$,..., $X_{n,n}$ be independent random variables such that

$$P[X_{i,n} = 1] = \frac{\lambda}{n}$$

$$P[X_{i,n} = 0] = 1 - \frac{\lambda}{n}.$$

Let $Y_n = X_{1,n} + X_{2,n} + ... + X_{n,n}$.

(a) (3 points) Find Φ_{Y_n}, the characteristic function of Y_n.

(b) (6 points) One can interpret Y_n as the number of successes in n independent Bernoulli trials with success probability λ/n. Verify that Y_n has a binomial distribution! Compute $\mathbb{E}[Y_n]$!

(c) (3 points) Find the limit of Φ_{Y_n} as $n \to \infty$. What distribution does it correspond to?
Hint: use the fact that $(1 + \frac{a}{n})^n$ converges to e^a as n tends to infinity.

33. (5 points) Suppose that

$$X_m \xrightarrow{\text{D}} X$$

and that there is a constant c such that $P[X = c] = 1$. Show that

$$X_m \xrightarrow{\text{i.p.}} X$$

[HINT: You need to show that $P[|X_m - X| > \epsilon] \to 0$ $(m \to \infty)$ for any ϵ. Because X is essentially constant and equal to c, this probability is equal to $1 - P[c - \epsilon \le X_m \le c + \epsilon]$ which is easily expressed in terms of the CDFs F_{X_m}. Now, use that F_{X_m} converges to F_X, which has a particularly simple form because X is essentially constant.]

34. (15 points total) Let X be a Bernoulli random variable which takes the values 1 and -1 both with probability $1/2$.

   (a) (3 points) Compute the characteristic function of X, i.e., $\phi_X(u) = \mathbb{E}[\exp(iuX)]$. Find a simple expression in terms of a cosine function.

   (b) (3 points) Verify that $\phi''(0) = -\mathbb{E}[X^2]$.

   (c) (4 points) Compute the characteristic function of

   $$Y_n = \frac{X_1 + \ldots + X_n}{2\sqrt{n}}$$

   using the above simple formula. Here, the random variables X_n are independent and of the same distribution as X.

   (d) (5 points) Approximate the cosine function by its Taylor polynomial of order 2 (i.e., the quadratic polynomial that provides the best approximation) and compute the limit of the characteristic function of Y_n. Conclude that Y_n converges in distribution. What is the limiting distribution?