

# ELEC 533 Homework 7

Due date: November 19, 2007  
54 points total (49 plus 5 bonus), 49 points = 100%  
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35. (8 points total) Recall the following inequality, which states that for a sequence of independent, identically distributed random variables  $X_n$  we can bound the probability of a deviation of sample means from the true mean as follows, in the case where  $\mathbb{E}[X] = 0$ :

$$P[(X_1 + \dots + X_n)/n > a] \leq \left( \frac{\mathbb{E}[e^{tX}]}{e^{ta}} \right)^n, t > 0$$

and that the minimum (or infimum) of the RHS, with respect to  $t$ , gives the tightest bound (the Chernoff bound), for a given specific distribution of the  $X_i, i = 1, \dots, n$ .

- (a) (3 points) Formulate a bound (the above inequality) in the general case, i.e., if  $\mathbb{E}[X] = \mu$ .  
(b) (5 points) Compute the tightest bound in the standard Gaussian case, i.e., for  $X_n \sim \mathcal{N}(0, 1)$ .
36. (7 points total) Let  $A$  and  $B$  be independent and identically distributed (iid) random variables with mean 0, variance  $\sigma^2$ , and third moment  $\mu = \mathbb{E}[A^3] = \mathbb{E}[B^3] \neq 0$ . Consider the random process

$$X(t) = A \cos(2\pi ft) + B \sin(2\pi ft) \quad -\infty < t < +\infty.$$

where  $f$  is a given frequency.

- (a) (3 points) Show that the random process  $X(t)$  is WSS.  
(b) (4 points) Show that  $X(t)$  is not strictly stationary.
37. (5 points) A discrete-time random process is WSS if  $E[X_n]$  does not depend on  $n$  and if the correlation  $E[X_n X_m]$  depends on  $n$  and  $m$  only through their difference. In this case,  $E[X_n X_m] = R_X(n - m)$ , where  $R_X$  is the univariate correlation function. Show that if  $X_n$  is WSS, then so is  $Y_n := X_n - X_{n-1}$ .
38. (15 points + 5 bonus points) Suppose  $X_i (i \in \mathbb{N})$  is a sequence of iid random variables and suppose

$$P[X_i = 1] = P[X_i = -1] = 1/2$$

for all  $i$ . Define a random process  $\{Y_t, t \in \mathbb{R}\}$  by

$$Y_t = X_i \quad \text{for all } t \text{ such that } i - 1 < t \leq i.$$

- (a) (3 points) Sketch a typical sample path of  $\{Y_t, t \in \mathbb{R}\}$ .  
(b) (6 points) Find the mean and autocorrelation function of  $\{Y_t, t \in \mathbb{R}\}$ . Is the process wide sense stationary (wss)? Show your argument.  
(c) (voluntary exercise; not required, 5 bonus points) Suppose  $A$  is a r.v. independent of the  $X_i$  and uniformly distributed in  $[0, 1]$ :  $A \sim \mathcal{U}([0, 1])$ . Define the random process  $\{Y_t, t \in \mathbb{R}\}$  by

$$Z_t = Y_{t+A}$$

Sketch a typical sample path of  $\{Z_t, t \in \mathbb{R}\}$ .

- (d) (6 points) Is  $Z_t$  (defined above) wss? Is  $Z_t$  second order stationary? Why or why not?

39. (9 points total) Let  $A$  and  $B$  be independent r.v. with means  $\mathbb{E}[A] = \mathbb{E}[B] = 0$  and variances  $\text{Var}(A) = \mathbb{E}[A^2] = \sigma_A^2$  and  $\text{Var}(B) = \mathbb{E}[B^2] = \sigma_B^2$ . Define the process  $\{X_t\}_{t \in \mathbb{R}}$  by

$$X_t = A \cos(t) + B \sin(t).$$

- (a) (3 points) Compute the autocorrelation  $R_X(t, s) = \mathbb{E}[X_t X_s]$ .
- (b) (6 points) Under what condition on  $\sigma_A$  and  $\sigma_B$  is  $X_t$  wide sense stationary (wss)? Give a full answer. HINT: Recall  $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ .
40. (5 points) Micrometeors strike the space shuttle according to a Poisson process. The expected time between strikes is 30 minutes. Find the probability that during at least one hour out of five consecutive hours (i.e., in at least one of the five consecutive 1-hour intervals) three or more micrometeors strike the shuttle.