ELEC 533 Homework 8

Due date: November 26, 2007

56 points total
Instructor: Dr. Erzsébet Merényi

41. (5 points) The Wiener process (or Brownian motion) $\{B_t\}$ is defined as the Gaussian process with zero mean and $R_B(t,s) = \sigma^2 \min(s,t)$. (In particular, $B_0 = 0$.) It has independent increments, more precisely, $B_t - B_s$ is independent of B_r provided $r \le t$ and $r \le s$. Define the process X_t as $X_t = B_t^2$. Compute the auto-correlation function R_X of X_t. (You may use that the 4-th moment of a $\mathcal{N}(0,\sigma^2)$-random variable is $3\sigma^4$.)

42. (18 points total) Let $\{X_t\}$ be a process given by the initial density f_{X_0} at time 0 and the conditional densities
$$f_{X_t|X_s=b}(a) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{1}{2}\frac{(a-b)^2}{t-s}\right)$$
for any $t > s$. The following questions are independent of each other and ordered in difficulty:
    (a) (2 points) Find the marginal f_{X_t} at time t if the initial density at time $s = 0$ is $P[X_0 = 1] = 1$. In other words, the process starts almost surely at $X_0 = 1$.
    (b) (4 points) Find the marginal f_{X_t} at time t if the initial density at time $s = 0$ is $X_0 \simeq \mathcal{N}(0,1)$. HINT: The solution involves an integral which is easy to solve noting that it is a convolution.
    (c) (5 points) Verify the Chapman-Kolmogorov equations. Hint: Convolution.
    (d) (7 points) Show that for fixed $s < t$ the random variables $X_t - X_s$ and X_s are independent.

43. (5 points total) *You already solved parts a) and b) of this problem in HW7. They are merely repeated here for completeness. Do not solve them again (hence the 0 points for them), but you may reuse any results from HW7.*

    Let A and B be independent r.v. with means $\mathbb{E}[A] = \mathbb{E}[B] = 0$ and variances $\mathrm{Var}(A) = \mathbb{E}[A^2] = \sigma_A^2$ and $\mathrm{Var}(B) = \mathbb{E}[B^2] = \sigma_B^2$. Define the process $\{X_t\}_{t\in\mathbb{R}}$ by
$$X_t = A\cos(t) + B\sin(t).$$
    (a) (0 points) Compute the autocorrelation $R_X(t,s) = \mathbb{E}[X_t X_s]$.
    (b) (0 points) Under what condition on σ_A and σ_B is X_t wide sense stationary (wss)? Give a full answer. HINT: Recall $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.
    (c) (5 points) Assume that A and B are Gaussian. Is X_t a Gaussian process? Show your argument.

44. (18 points total) Let $\{X_t\}$ be a continuous time, but discrete valued Markov process. For simplicity we assume that the values are integers. As in the case with continuous values treated in class, we define the process via its initial distribution $P[X_0 = k]$ at time 0 and the transition probabilities
$$P[X_t = k | X_s = n]$$
for any $t > s$. More precisely, the marginals are
$$P[X_t = k] = \sum_{n=1}^{\infty} P[X_t = k | X_0 = n] \cdot P[X_0 = n]$$

and the f.d.d. are

$$P[X_{t_1} = k_1, \dots X_{t_n} = k_n] = P[X_{t_n} = k_n | X_{t_{n-1}} = k_{n-1}] \cdots P[X_{t_2} = k_2 | X_{t_1} = k_1] \cdot P[X_{t_1} = k_1]$$

As in the continuous-valued case done in class, this defines a consistent f.d.d. if and only if the Chapman-Kolmogorov equations are satisfied. Here, these Chapman-Kolmogorov equations read as:

$$\sum_{n=1}^{\infty} P[X_t = k | X_s = n] \cdot P[X_s = n | X_r = m] = P[X_t = k | X_r = m].$$

(a) (6 points) Check that the Chapman-Kolmogorov equations imply consistency. To this end, assume that the Chapman-Kolmogorov equations hold and show that for any $j = 1, \dots n$

$$\sum_{k_j=1}^{\infty} P[X_{t_1} = k_1, \dots X_{t_n} = k_n] = P[X_{t_1} = k_1, \dots, X_{t_{j-1}} = k_{j-1}, X_{t_{j+1}} = k_{j+1}, \dots, X_{t_n} = k_n].$$

(b) (7 points) Consider the following particular transition probability where $t > s$ and where k and n are positive integers:

$$P[X_t = k | X_s = n] = \begin{cases} 0 & \text{if } k < n \\ e^{-\lambda(t-s)} (\lambda(t-s))^{k-n} / (k-n)! & \text{if } k \geq n. \end{cases}$$

In other words, given that $X_s = n$ the *increment* $X_t - X_s$ is a Poisson random variable of mean $\lambda(t-s)$.
Verify that the Chapman-Kolmogorov equations are satisfied. Hint 1: recall that

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$

Hint 2: be careful about noting when $P[X_t = k | X_s = n]$ is zero.

(c) (5 points) With the same transition probabilities as before add the initial distribution $P[X_0 = 0] = 1$. Show that X_t is a Poisson variable of mean λt.

Note: In this case, it is possible to show that X_t is actually a Poisson renewal process. This is not obvious, since one needs to construct arrival times T_k of events which should be spaced by independent waiting times. A first step towards this goal would be to show that the increments $X_t - X_s$ over disjoint time intervals are independent.

45. (10 points total) *This problem is part of Problem 7.19 from Stark and Woods, page 477. It has a figure, for which you are referred to the book. If you do not have the book please let us know and we will give you the figure.*

Consider the three-state Markov chain $N(t)$ with the state-transition diagram shown in Figure P7.19. Here the state labels are the actual outputs, i.e., $N(t) = 3$ while the chain is in state 3. The state transitions are governed by jointly independent, exponentially distributed interarrival times, with average rates as indicated on the branches.

(a) (5 points) Write the differential equations for the probability of being in a state i at time $t \geq 0$, denoting the probabilities as $p_i(t), i = 1, 2, 3$. [Help: see Example with Good / Bad states from class, and read S & W, pages 423 – 427.]

(b) (5 points) Find the steady-state solution for $p_i(t), i = 1, 2, 3$, i.e., $p_i(\infty)$.