ELEC 533 Homework 9

Due date: December 3, 2007

62 points total (60 points = 100%)
Instructor: Dr. Erzsébet Merényi

46. (20 points total) Throughout let all processes be wss. Recall: for X_t the power spectral density $S_X(\nu)$ is defined as the Fourier transform of the auto-correlation function $R_X(\tau) = \mathbb{E}[X_{s+\tau} \cdot X_s^*]$.

    (a) (3 points) Let U_t and V_t be orthogonal wss processes, meaning that $\mathbb{E}[U_t V_s^*] = 0$ for all t and s. Set $W_t = U_t + V_t$. Show that $R_W(\tau) = R_U(\tau) + R_V(\tau)$ and conclude that $S_W(\nu) = S_U(\tau) + S_V(\tau)$.

    (b) (2 points) If $Y_t = aX_t$ then $R_Y(\tau) = |a|^2 R_X(\tau)$ and $S_Y(\nu) = |a|^2 S_X(\nu)$.

    (c) (4 points) Let $Y_t = X_t \cdot \cos(\nu_0 t + \Theta)$ where ν_0 is a fixed frequency and where the random phase Θ is independent of the process X_t and uniformly distributed on $[-\pi, \pi]$. Show that $R_Y(\tau) = 1/2 R_X(\tau)\cos(\nu_0 \tau)$ and $S_Y(\nu) = 1/4(S_X(\nu + \nu_0) + S_X(\nu - \nu_0))$.

    (d) (3 points) Assume that X_t be a zero mean process, meaning that $\mu_X(t) = 0$. Set $Z_t = X_t + b$. Show that $R_Z(\tau) = R_X(\tau) + |b|^2$ and $S_Z(\nu) = S_X(\nu) + 2\pi|b|^2\delta(\nu)$.

    (e) (4 points) Assume that A_t is a wss process such that its auto-correlation function is related to that of X_t via $R_A(\tau) = -R''_X(\tau)$. Show that $S_A(\nu) = \nu^2 S_X(\nu)$. (Here, R'' denotes the second derivative.)

    (f) (4 points) Determine whether or not $R(\tau) = \frac{1}{(1+\tau^2)}$ is a valid correlation function.

    Most of these questions can be derived from properties of the Fourier transform.

47. (6 points total) Let the random process $Y(t)$ be given as

    $$Y(t) = X(t) + 0.3\frac{dX(t)}{dt},$$

    where $X(t)$ is a random process with the mean function $\mu_X(t) = 5t$, and covariance function

    $$K_{XX}(t_1, t_2) = \frac{\sigma^2}{1 + \alpha(t_1 - t_2)^2}, \qquad \alpha > 0.$$

    (a) (2 points) Find the mean function $\mu_Y(t)$

    (b) (3 points) Find the covariance function $K_{YY}(t_1, t_2)$

    (c) (1 point) Is the random process $Y(t)$ wide-sense stationary (WSS)? Why?

48. (10 points) Let $X(t)$ be a random process defined by

    $$X(t) = N\cos(2\pi f_0 t + \Theta),$$

    where f_0 is a known frequency and N and Θ are independent random variables. The characteristic function for N is

    $$\Phi_N(\omega) = \mathbb{E}[e^{+j\omega N}] = \exp\{\lambda[e^{jw} - 1]\},$$

    where λ is a given constant (that is, N is a Poisson random variable). The random variable Θ is uniformly distributed on $[-\pi, +\pi]$.

    (a) (1 point) Determine the mean function $\mu_X(t)$.

(b) (4 points) Determine the covariance function $K_{XX}(t,s)$.

(c) (1 point) Is $X(t)$ WSS? Justify your answer.

(d) (4 points)Is $X(t)$ stationary? Justify your answer.

49. (8 points) Let the random process $X(t)$ be wide-sense stationary (WSS) with correlation function

$$R_{XX}(\tau) = \sigma^2 e^{-(\tau/T)^2}.$$

Let $Y(t) = 3X(t) + 2X'(t)$, where the derivative is interpreted in the mean-square (m.s.) sense.

(a) (3 points) State conditions for the existence of such a $Y(t)$ in terms of a *general* correlation function $R_{XX}(\tau)$.

(b) (5 points) Find the correlation function $R_{YY}(\tau)$ for the given $R_{XX}(\tau)$ in terms of σ^2 and T.

50. (10 points) Let X_t be a random process with constant mean $\mu_X \neq 0$ and covariance function $K_{XX}(t,s) = \sigma^2 \cos(\nu_0(t-s))$.

(a) (5 points) Show that the m.s. derivative X_t' exists.

(b) (3 points) Find the covariance function $K_{X'X'}(t,s)$ of the m.s. derivative X_t'.

(c) (2 points) Find the correlation function $R_{X'X'}(t,s)$ of the m.s. derivative X_t'.

51. (8 points) Let $X(t)$ be a stationary random process with mean μ_X and covariance function

$$K_{XX}(\tau) = \frac{\sigma_X^2}{1+\alpha^2\tau^2}, \quad -\infty < \tau < +\infty.$$

(a) (5 points) Show that a mean-square derivative exists for all t.

(b) (3 points) Find $\mu_{\dot{X}}(t)$ and $K_{\dot{X}\dot{X}}(\tau)$ for all t and τ.

The following is for voluntary exercise, to further help your preparation for the final. You do not have to turn this in, and do not receive points for it. (Points are shown at each subproblem to indicate their relative level of difficulty). We will, however, provide the solution along with the solutions to the rest of the problems in this HW.

52. (5 points) Let $R_0(\tau)$ be a correlation function, and define $R(\tau) := R_0(\tau)cos(2\pi f_0 \tau)$ for some $f_0 > 0$. Determine whether or not $R(\tau)$ is a valid correlation function.

53. (5 points) The power spectral density (psd) of a random process $\{X_t\}_T, T = \mathbb{R}$, is given as

$$S_X(\nu) = a + b\cos(2\pi\nu)$$

for some constants a and b. What values of a and b will result in a valid psd?

54. (5 points) Compute the spectral density for processes with the following auto-correlation: $R_X(\tau) = \max(1 - |\tau|/T, 0)$. HINT: R_X is the convolution of two extremely simple functions with extremely simple Fourier transforms.

55. (20 points total) Suppose A and B are independent Gaussian random variables, i.e. $A \sim \mathcal{N}(0,1)$ and $B \sim \mathcal{N}(0,1)$. Define the random process $\{X_t, t \in \mathbb{R}\}$ by

$$X_t = A + Bt + t^2.$$

(a) (8 points) Compute the following marginal distributions: ϕ_{X_t} for arbitrary t and ϕ_{X_1,X_5}.

(b) (4 points) Are $X_1, X_2, \ldots, X_5$ jointly Gaussian?

(c) (4 points) Find $\mathbb{E}[X_5 \mid X_0]$. This is a prediction of X_5 knowing X_0, and it is a random variable. Compute its variance.

(d) (4 points) Find $\mathbb{E}[X_5 \mid X_0, X_1]$. This is a prediction of X_5 knowing X_0 and X_1, and it is also a random variable. Compute its variance.