Problem 5.1

(a) The function \( g(x) = e^{-2x} \) is strictly monotonically decreasing (p 184), and so we can find its inverse \( h(x) = \frac{\ln(x)}{2} \). Using the formula from page 185, we know the expression for the PDF of \( Y \) over its support is given by:

\[
  f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|
\]

\[
  = \frac{1}{2y}
\]

We can find the support by inspection, and so the PDF is given by:

\[
  f_Y(y) = \begin{cases} 
    \frac{1}{2y} & , \quad e^{-4} \leq y \leq e^{-2} \\
    0 & , \quad \text{otherwise}
  \end{cases}
\]

(b) This function is not strictly monotonic. However, we can compute the PDF via a CDF expression:

\[
  \Pr(Y \leq y) = \Pr((X - 1.2)^2 \leq y)
  = \Pr(-\sqrt{y} + 1.2 \leq X \leq \sqrt{y} + 1.2)
  = F_X(\sqrt{y} + 1.2) - F_X(-\sqrt{y} + 1.2)
  = \begin{cases} 
    2\sqrt{y} & , \quad 0 \leq y \leq 0.04 \\
    \sqrt{y} + 0.2 & , \quad 0.04 \leq y \leq 0.64 \\
    0 & , \quad y < 0 \\
    1 & , \quad y > 0.64
  \end{cases}
\]

Taking derivatives, we have the expression for the PDF of \( Y \):

\[
  f_Y(y) = \begin{cases} 
    \frac{1}{\sqrt{y}} & , \quad 0 \leq y \leq 0.04 \\
    \frac{1}{2\sqrt{y}} & , \quad 0.04 < y \leq 0.64 \\
    0 & , \quad \text{otherwise}
  \end{cases}
\]
Problem 5.1

(a) Because \( X \) and \( Y \) are independent, we know \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \). Thus, the joint pdf is

\[
f_{X,Y}(x,y) = \begin{cases} 2e^{-2y} & 0 \leq x \leq 1, y \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

\( P(Y \geq X) \) is simply \( 1 - P(Y \leq X) \). The region where \( Y \leq X \) is a triangle, where \( x \) is from 0 to 1 and \( y \) is from 0 to \( x \).

\[
P(Y \geq X) = \int_{y=x} \int f_{X,Y}(x,y) \, dx \, dy
\]

\[
= 1 - \int_0^1 \int_0^x 2e^{-2y} \, dy \, dx
\]

\[
= 1 - \int_0^1 1 - e^{-2x} \, dx
\]

\[
= \frac{1}{2} - \frac{e^{-2}}{2}
\]

(b) Conditioning on \( Y = y \), find the PDF of \( Z \)

\[
f_{Z|Y=y}(z) = f_{X+Y|Y=y}(x+y)
\]

\[
= \begin{cases} 1 & y \leq z \leq 1 + y \\ 0 & \text{otherwise} \end{cases}
\]

(c)

\[
F_Z(z) = \int_0^1 \int_0^{z-x} f_{X,Y}(x,y) \, dy \, dx
\]

\[
= \int_0^1 \int_0^{z-x} 2e^{-2y} \, dy \, dx
\]

\[
= \int_0^1 [-e^{-2y}]_{0}^{x} \, dx
\]

\[
= \int_0^1 1 - e^{-2(z-x)} \, dx
\]

\[
= x - \frac{e^{-2(z-x)}}{2} \bigg|_0^1
\]

\[
= 1 - \frac{e^{-2z+2}}{2} + \frac{e^{-2z}}{2}
\]

Thus, \( f_Z(z) = -e^{-2z} + e^{-2z+2} \).
Problem 5.3

(a) Let $f_A(a)$ represent the PDF of the time Alice spends to complete her problem set. Let $f_B(b)$ represent the PDF of the time Bob spends to complete his problem set. Since Alice and Bob work independently on their problem sets, we know the following:

$$f_{A,B}(a,b) = f_A(a) \cdot f_B(b) = \left(\frac{1}{4}e^{-\frac{a}{4}}\right) \cdot \left(\frac{1}{6}e^{-\frac{b}{6}}\right) = \frac{1}{24}e^{-\frac{6a+4b}{24}}$$

Now compute $P(A < B)$, the probability that Alice finishes her homework before Bob.

$$P(A < B) = \int_0^\infty \int_0^b f_{A,B}(a,b)\,dadb = \frac{1}{24} \int_0^\infty \int_0^b e^{-\frac{6a+4b}{24}}\,dadb$$

$$= \frac{1}{24} \int_0^\infty \left[-4e^{-\frac{a}{4}}\right]_{a=0}^{a=b} db = \frac{1}{6} \int_0^\infty e^{-\frac{b}{6}} \left(1 - e^{-\frac{4}{5}}\right) \, db$$

$$= \frac{1}{6} \left[-6e^{-\frac{b}{6}} + \frac{12}{5} e^{-\frac{2b}{5}}\right]_{b=0}^{b=\infty} = 3$$

(b) Let us compute $P(A < B|A > 4)$ which is the probability that Alice finishes the problem set before Bob given that Alice requires more than 4 hours. Using the definition of conditional probability this can be computed as follows.

$$P(A < B|A > 4) = \frac{P(4 < A < B)}{P(A > 4)} = \frac{\int_4^\infty \int_4^b f_{A,B}(a,b)\,dadb}{\int_4^\infty f_A(a)\,da}$$

$$= \frac{\frac{1}{24} \int_4^\infty \int_4^b e^{-\frac{6a+4b}{24}}\,dadb}{\frac{1}{4} \int_4^\infty e^{-\frac{4}{5}}\,da} = \frac{3}{5} e^{-\frac{4}{3}}$$

(c) The question asks for the following probability which can be broken up into two probabilities since the two cases are disjoint.

$$P(\text{difference in finish times} > 1\,\text{hour}) = P(A > B + 1) + P(A < B - 1)$$
Problem 5.4.

Now let us compute these probabilities.

\[
P(A > B + 1) = \int_0^\infty \int_{b+1}^{\infty} f_{A,B}(a,b) \, da \, db
\]
\[
= \frac{1}{24} \int_0^\infty \int_{b+1}^{\infty} e^{-\frac{a+b}{24}} \, da \, db
\]
\[
= \frac{2}{5} e^{-\frac{1}{4}}
\]

\[
P(A < B - 1) = \int_1^\infty \int_0^{b-1} f_{A,B}(a,b) \, da \, db
\]
\[
= \frac{1}{24} \int_1^\infty \int_0^{b-1} e^{-\frac{a+b}{24}} \, da 
\]
\[
= \frac{3}{5} e^{-\frac{1}{6}}
\]

Therefore the answer is \(\frac{2}{5} e^{-\frac{1}{4}} + \frac{3}{5} e^{-\frac{1}{6}}\)

Problem 5.4.

(a) Since \(X\) and \(Y\) are independent, the probability that the maximum of them is less than or equal to \(c\) equals the product of the probability of each being less than or equal to \(c\). Therefore, for \(c \geq 0\), we have,

\[
F_{\max}(X,Y)(c) = P(X \leq c, Y \leq c) = P(X \leq c)P(Y \leq c) = F_X(c)F_Y(c)
\]
\[
= 1 - e^{-\lambda_X c} - e^{-\lambda_Y c} + e^{-(\lambda_X + \lambda_Y)c}
\]

The CDF is zero for \(c < 0\). The pdf of \(\max(X,Y)\) is

\[
f_{\max}(X,Y) = F'_{\max}(X,Y) = \lambda_X e^{-\lambda_X c} + \lambda_Y e^{-\lambda_Y c} - (\lambda_X + \lambda_Y)e^{-c(\lambda_X + \lambda_Y)}, \quad c \geq 0.
\]

In all,

\[
f_{\max}(X,Y) = \begin{cases} 
\lambda_X e^{-\lambda_X c} + \lambda_Y e^{-\lambda_Y c} - (\lambda_X + \lambda_Y)e^{-c(\lambda_X + \lambda_Y)}, & c \geq 0. \\
0, & c < 0.
\end{cases}
\]
To find the pdf of \( C = X + Y \), we need to sum the two dimensional probability over the shaded area.

\[
F_C(c) = P(C \leq c) = \int_0^c \int_0^{c-x} f_{X,Y}(x,y) \, dy \, dx = \int_0^c \int_0^{c-x} f_X(x)f_Y(y) \, dy \, dx \\
= \int_0^c \int_0^{c-x} \lambda_x e^{-\lambda x} \lambda_y e^{-\lambda y} \, dy \, dx \\
= \left\{ \begin{array}{ll} 
1 - \frac{\lambda_x e^{-\lambda x} - \lambda_y e^{-\lambda y}}{\lambda_x - \lambda_y}, & \lambda_x \neq \lambda_y \\
1 - (1 + c\lambda)e^{-c\lambda}, & \lambda_x = \lambda_y = \lambda.
\end{array} \right.
\]

Then, for \( c \geq 0 \), the pdf is

\[
f_C(c) = F_C'(c) = \left\{ \begin{array}{ll} 
\lambda_x \lambda_y \frac{e^{-\lambda y} - e^{-\lambda x}}{\lambda_x - \lambda_y}, & \lambda_x \neq \lambda_y \\
\frac{c\lambda^2 e^{-c\lambda}}{c\lambda^2 e^{-c\lambda}}, & \lambda_x = \lambda_y = \lambda.
\end{array} \right.
\]