ELEC 303 – Random Signals

Lecture 1 - General info, Sets and Probabilistic Models
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General information

• Syllabus/policy handout
• Course webpage:
  http://www.ece.rice.edu/~fk1/classes/ELEC303.htm
• Recommended books on webpage
• Instructor and TA office hours on the webpage
Grading / policy

• Grading
  – Quiz 1: 10%
  – Midterm: 25%
  – Quiz 2: 10%
  – Final: 25%
  – Homework and matlab assignments: 25%
  – Participation: 5%
• Read the policy handout for the homework policy, cheating policy, and other information of interest

Lecture outline

• Reading: Sections 1.1, 1.2
• Motivation
• Sets
• Probability models
  – Sample space
  – Probability laws – axioms
  – Discrete and continuous models
Motivation

• What are random signals and probability?
• Can we avoid them?

• Why are they useful?
• What are going to learn?

Sets – quick review

• A set (S) is a collection of objects \( (x_i) \) which are the elements of S, shown by \( x_i \in S, i=1,\ldots,n \)
• S may be finite or countably infinite
Sets – quick review

- Set operations and notations:
  - Universal set: $\Omega$, empty set: $\emptyset$, complement: $S^c$
  - Union: $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \ldots = \{x \mid x \in S_n \text{ for some } n\}$
  - Intersection: $\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \ldots = \{x \mid x \in S_n \text{ for all } n\}$
  - De Morgan’s laws:
    1. $(\bigcup_{n} S_n)^c = \bigcap_{n} S_n^c$
    2. $(\bigcap_{n} S_n)^c = \bigcup_{n} S_n^c$

Venn diagram

- A representation of sets
Probabilistic models

- **Sample space**: set of all possible outcomes of an experiment (mutually exclusive, collectively exhaustive)
- **Probability law**: assigns to a set A of possible outcomes (events) a nonnegative number \( P(A) \)
  - \( P(A) \) is called the **probability** of A

![Diagram of experiment and event space](image)

From frequency to probability (1)

The time of recovery (Fast, Slow, Unsuccessful) from an ACL knee surgery was seen to be a function of the patient’s age (Young, Old) and weight (Heavy, Light).

The medical department at MIT collected the following data:

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<tbody>
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<td>1000</td>
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<td>O</td>
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![Graph showing frequency distribution](image)
From frequency to probability (2)

<table>
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<th></th>
<th>Y, L</th>
<th>Y, H</th>
<th>O, L</th>
<th>O, H</th>
</tr>
</thead>
<tbody>
<tr>
<td>S, Fast</td>
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<td>500</td>
<td>400</td>
<td>200</td>
</tr>
<tr>
<td>S, Slow</td>
<td>1500</td>
<td>300</td>
<td>400</td>
<td>600</td>
</tr>
<tr>
<td>U</td>
<td>50</td>
<td>100</td>
<td>200</td>
<td>300</td>
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</table>

- What is the likelihood that a 40 years old man (old!) will have a successful surgery with a speedy recovery?
- If a patient undergoes operation, what is the likelihood that the result is unsuccessfull?
- Need a measure of “likelihood”
- Ingredients: sample space, events, probability

Sequential models – sample space

Two rolls of a tetrahedral die
- Sample space vs. sequential description
Axioms of probability

1. **(Nonnegativity)** $0 \leq P(A) \leq 1$ for every event $A$
2. **(Additivity)** If $A$ and $B$ are two disjoint events, then the probability
   \[ P(A \cup B) = P(A) + P(B) \]
3. **(Normalization)** The probability of the entire sample space $\Omega$ is equal to 1, i.e., $P(\Omega) = 1$

Discrete models

- **Example:** coin flip – head (H), tail (T)
- Assume that it is a fair coin
- What is the probability of getting a T?
- What is the probability of getting 2 H’s in three coin flips?
- **Discrete probability law** for a finite number of possible outcomes: the probability of an event is the sum of it’s disjoint elements’ probabilities
Example: tetrahedral dice

- Let every possible outcome have probability 1/16
- \( P(X=1) = P(X=2) = P(X=3) = P(X=4) = 0.25 \)
- Define \( Z = \min(X,Y) \)
- \( P(Z=1) = ? \)
- \( P(Z=2) = ? \)
- \( P(Z=3) = ? \)
- \( P(Z=4) = ? \)

**Discrete uniform law:**
Let all sample points be equally likely, then

\[
P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}}
\]

Continuous probability

- Each of the two players randomly chose a number in \([0,1]\). What is the probability that the two numbers are at most \( \frac{1}{4} \) apart?
- Draw the sample space and event area
- Choose a probability law
  - Uniform law: probability = area
Some properties of probability laws

- If $A \subseteq B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) \leq P(A) + P(B)$
- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

Models and reality

- Probability is used to model uncertainty in real world
- There are two distinct stages:
  - Specify a probability law suitably defining the sample space. No hard rules other than the axioms
  - Work within a fully specified probabilistic model and derive the probabilities of certain events