ELEC 303 – Random Signals

Lecture 13 – Transforms

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Lecture outline

• Reading: 4.4-4.5
• Definition and usage of transform
• Moment generating property
• Inversion property
• Examples
• Sum of independent random variables
Transforms

- The **transform** for a RV $X$ (a.k.a **moment generating function**) is a function $M_X(s)$ with parameter $s$ defined by $M_X(s) = E[e^{sx}]$

**Discrete, PMF:**

$$M_X(s) = E[e^{sx}] = \sum_x e^{sx} p_X(x)$$

**Continuous, PDF:**

$$M_X(s) = E[e^{sx}] = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

Why transforms?

- New representation
- Has usages in
  - Calculations, e.g., moment generation
  - Theorem proving
  - Analytical derivations
Example(s)

- Find the transforms associated with
  - Poisson random variable
  - Exponential random variable
  - Normal random variable
  - Linear function of a random variable

Moment generating property

- Moments: $E[X^n]$, we need to integrate.
- Can instead differentiate the transform:

$$M_X(s) = E[e^{sX}] = \left\{ \begin{array}{ll}
\sum_{i=0}^{\infty} p_X(i) e^{si} & \text{if } s \neq 0 \\
\int_{-\infty}^{\infty} e^{sx} f_X(x) \, dx & \text{if } s = 0
\end{array} \right.$$  

$$M_X(s) \bigg|_{s=0} = E[e^{0X}] = 1$$  

$$\frac{d}{ds} M_X(s) \bigg|_{s=0} = E[X]$$  

$$\frac{d^n}{ds^n} M_X(s) \bigg|_{s=0} = E[X^n]$$
Example

• Use the moment generation property to find the mean and variance of exponential RV

\[ f_X(x) = \lambda e^{-\lambda x} \text{ over } x \geq 0 \ (\lambda > 0) \]

Inverse of transforms

• Transform \( M_X(s) \) is invertible – important

  The transform \( MX(s) \) associated with a RV \( X \) uniquely determines the CDF of \( X \), assuming that \( MX(s) \) is finite for all \( s \) in some interval \([-a,a]\), \( a>0 \)

• Explicit formulas that recover PDF/PMF from the associated transform are difficult to use

• In practice, transforms are often converted by pattern matching
Inverse transform – example 1

• The transform associated with a RV X is
• \( M(s) = \frac{1}{4} e^{-s} + \frac{1}{2} + \frac{1}{8} e^{4s} + \frac{1}{8} e^{5s} \)
• We can compare this with the general formula
• \( M(s) = \sum_x e^{sx} p_X(x) \)
• The values of X: -1, 0, 4, 5
• The probability of each value is its coefficient
• PMF: \( P(X=-1)=\frac{1}{4}; \quad P(X=0)=\frac{1}{2}; \quad P(X=4)=\frac{1}{8}; \quad P(X=5)=\frac{1}{8}; \)

Inverse transform – example 2

• If we know \( X \) takes nonnegative integer values:
\[
M_X(s) = E[e^{sX}] = \sum_x e^{sx} p_X(x) \\
= p_X(0) + p_X(1)e^s + p_X(2)e^{2s} + \cdots
\]
• Now, say we have: \( M_X(s) = \frac{pe^s}{1 - (1-p)e^s} \)
• Recall: \( \frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 - \cdots \) for \(|\alpha| < 1\)
• So: \( M_X(s) = pe^s \left(1 + (1-p)e^s + (1-p)^2e^{2s} + \cdots\right)\)
• We recognize: \( p_X(x) = p(1-p)^{x-1} \) for \( x = 1, 2, \ldots \)

This is the geometric PMF.
Mixture of two distributions

- Example: \( f_X(x) = \frac{2}{3} \cdot 6e^{-6x} + \frac{1}{3} \cdot 4e^{-4x}, \quad x \geq 0 \)
- More generally, let \( X_1, \ldots, X_n \) be continuous RV with PDFs \( f_{X_1}, f_{X_2}, \ldots, f_{X_n} \)
- Values of RV \( Y \) are generated as follows
  - Index \( i \) is chosen with corresponding prob \( p_i \)
  - The value \( y \) is taken to be equal to \( X_i \)
    \[
    f_Y(y) = p_1 f_{X_1}(y) + p_2 f_{X_2}(y) + \ldots + p_n f_{X_n}(y)
    \]
    \[
    M_Y(s) = p_1 M_{X_1}(s) + p_2 M_{X_2}(s) + \ldots + p_n M_{X_n}(s)
    \]
- The steps in the problem can be reversed then

Sum of independent RVs

- \( X \) and \( Y \) independent RVs, \( Z = X + Y \)
- \( M_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}] \)
  \[
  = E[e^{sX}]E[e^{sY}] = M_X(s) M_Y(s)
  \]
- Similarly, for \( Z = X_1 + X_2 + \ldots + X_n \),
  \[
  M_Z(s) = M_{X_1}(s) M_{X_2}(s) \ldots M_{X_n}(s)
  \]
Sum of independent RVs – example 1

- $X_1, X_2, ..., X_n$ independent Bernouli RVs with parameter $p$
- Find the transform of $Z = X_1 + X_2 + ... + X_n$

Sum of independent RVs – example 2

- $X$ and $Y$ independent Poisson RVs with means $\lambda$ and $\mu$ respectively
- Find the transform for $Z = X + Y$
- Distribution of $Z$?
Sum of independent RVs – example 3

- X and Y independent normal RVs
- \( X \sim N(\mu_x, \sigma_x^2) \), and \( Y \sim N(\mu_y, \sigma_y^2) \)
- Find the transform for \( Z=X+Y \)
- Distribution of \( Z \)?
Bookstore example (1)

- George visits a number of book stores looking for the "Hair Book".
- A bookstore carries such a book with probability \( \frac{1}{3} \).
- The time George spends in each book store is exponentially distributed with \( \lambda = 3 \).
- George will visit bookstores until he finds the book.
- We want to find the PDF, mean, variance of the time he spends in bookstores.
- **Total time:** \( Y = X_1 + X_2 + \cdots + X_N \)

Sum of random number of independent RVs

- \( N \): nonnegative integer-valued r.v.
- \( X_1, X_2, \ldots \): i.i.d. r.v.s, independent of \( N \).
- Let: \( Y = X_1 + \cdots + X_N \). Then:
  - **Mean:** \( \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N\mathbb{E}[X]] = \mathbb{E}[N]\mathbb{E}[X] \)
  - **Variance:** \( \text{Var}(Y) = \mathbb{E}[\text{Var}(Y|N)] + \text{Var}(\mathbb{E}[Y|N]) = \mathbb{E}[N]\text{Var}(X) + (\mathbb{E}[X])^2\text{Var}(N) \)
Bookstore example (2)

- Number of bookstores, \( N \):
  - PMF \( p_N(n) = \left( \frac{1}{3} \right)^n \) \( \left( \text{geometric, from } n=1 \right) \)
  - Mean \( \mathbb{E}[N] = \frac{1}{\frac{1}{3}} = 3 \)
  - Variance \( \text{Var}(N) = \frac{1}{\left( \frac{1}{3} \right)^2} = 6 \)

- Time in each bookstore, \( X \) (i.i.d., indep of \( N \)):
  - PDF \( f_X(x) = 3e^{-3x} \) \( x \geq 0 \)
  - Mean \( \mathbb{E}[X] = \frac{1}{3} \)
  - Variance \( \text{Var}(X) = \frac{1}{3} \)

- Total time, \( Y \):
  - Mean \( \mathbb{E}[Y] = \mathbb{E}[N]\mathbb{E}[X] = 1 \)
  - Variance \( \text{Var}(Y) = \mathbb{E}[N]\text{Var}(X) + (\mathbb{E}[X])^2\text{Var}(N) \)
    \( = 1 \)

Transform of random sum

- \( N \): nonnegative integer-valued r.v.
- \( X_1, \ldots, X_N \): i.i.d. r.v.s, independent of \( N \).
- If \( Y = X_1 + \cdots + X_N \), we have:

\[
M_Y(s) = \mathbb{E}[e^{sY}]
= \mathbb{E}[\mathbb{E}[e^{sY}|N]]
= \mathbb{E}[\mathbb{E}[e^{s(X_1+\cdots+X_N)|N}]]
= \mathbb{E}[M_X(s)^N]
\]

- Compare with: \( M_N(s) = \mathbb{E}[(e^s)^N] \)

Thus, to get \( M_Y(s) \), start with \( M_N(s) \) and replace each occurrence of \( e^s \) by \( M_X(s) \).
Bookstore example (3)

- Number of bookstores:
  - Transform $M_N(s) = \frac{c^{N/3}}{1 - 2e^{s/3}} = E(e^{s})^N$.
- Time in each bookstore:
  - Transform $M_X(s) = \frac{3}{3 - s}$.
- Total time:
  - Transform $M_Y(s) = E[M_X(s)^N]$
    $$= \frac{\left(\frac{3}{s}ight)^{N/3}}{1 - 2\left(\frac{2}{3-s}\right)^{N/3}} = \frac{1}{1 - s}$$
  - PDF: $f_Y(y) = e^{-y}$ $y \geq 0$ (exponential, with $\lambda = 1$).

More examples...

- A village with 3 gas station, each open daily with an independent probability 0.5
- The amount of gas in each is $\sim U[0,1000]$
- Characterize the probability law of the total amount of available gas in the village
More examples...

• Let $N$ be Geometric with parameter $p$
• Let $X_i$’s Geometric with common parameter $q$
• Find the distribution of $Y = X_1 + X_2 + \ldots + X_n$