Total probability theorem

- Divide and conquer
- Partition the sample space into $A_1$, $A_2$, $A_3$
- For any even $B$:
  \[ P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \]
  \[ = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \]
Radar example 3 (cont’d)

<table>
<thead>
<tr>
<th>Radar Airplane</th>
<th>Low(0)</th>
<th>Medium(?)</th>
<th>High(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absent</td>
<td>0.45</td>
<td>0.20</td>
<td>0.05</td>
</tr>
<tr>
<td>Present</td>
<td>0.02</td>
<td>0.08</td>
<td>0.20</td>
</tr>
</tbody>
</table>

- \( P(\text{Present}) = 0.3 \)
- \( P(\text{Medium} | \text{Present}) = 0.08/0.3 \)
- \( P(\text{Present} | \text{low}) = 0.02/0.47 \)

Given the radar reading, what is the best decision about the plane?

- Criterion for decision: minimize “probability of error”
- Decide **absent** or **present** for each reading
Radar example 3 (cont’d)

- Error={Present and decision is absent} or {Absent and decision is present}
- Disjoint events
- \( P(\text{error}) = 0.02 + 0.08 + 0.05 \)

Extended radar example

- Threat alert affects the outcome

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{Radar} & \text{Low(0)} & \text{Medium(?)} & \text{High(1)} \\
\hline
\text{Absent} & 0.1125 & 0.05 & 0.0125 \\
\text{Present} & 0.055 & 0.22 & 0.55 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
& \text{Radar} & \text{Low(0)} & \text{Medium(?)} \\
\hline
\text{Absent} & 0.45 & 0.20 & 0.05 \\
\text{Present} & 0.02 & 0.08 & 0.20 \\
\hline
\end{array}
\]

- \( P(\text{Threat}) = \text{Prior probability of threat} = p \)
Extended radar example

- A = Airplane, R = Radar reading
  \[ P(A, R) = P(\text{Threat})P(A, R|\text{Threat}) + P(\text{No Threat})P(A, R|\text{No Threat}) \]

- If we let \( p = P(\text{Threat}) \), then we get:

<table>
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<tr>
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<th>Medium(?)</th>
<th>High(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absent</td>
<td></td>
<td>0.45-0.3375(p)</td>
<td>0.20-0.15(p)</td>
<td>0.05-0.0375(p)</td>
</tr>
<tr>
<td>Present</td>
<td></td>
<td>0.02+0.014(5p)</td>
<td>0.08+0.14(p)</td>
<td>0.20+35(p)</td>
</tr>
</tbody>
</table>

- Given the radar registered high and a plane was absent, what is the probability that there was a threat?
- How does the decision region behave as a function of \( p \)?
Extended radar example

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<th>Medium(?)</th>
<th>High(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absent</td>
<td>0.45-</td>
<td>0.3375p</td>
<td>0.05-</td>
</tr>
<tr>
<td></td>
<td>0.375p</td>
<td></td>
<td>0.0375p</td>
</tr>
<tr>
<td>Present</td>
<td>0.02+0.014</td>
<td>0.08+0.14p</td>
<td>0.20+35p</td>
</tr>
</tbody>
</table>

\[ P(T|\text{High}&\text{Absent}) = \frac{.0125p}{.05 - .0375p} \]

Check \( p = .5 \)?

Bayes rule

- The total probability theorem is used in conjunction with the Bayes rule:

\[
P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}
\]

Let \( A_1, A_2, \ldots, A_n \) be disjoint events that form a partition of the sample space, and assume that \( P(A_i) > 0 \), for all \( i \). Then, for any event \( B \) such that \( P(B) > 0 \), we have
Bayes rule inference

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + \ldots + P(A_3)P(B \mid A_3)} \quad i = 1, 2, 3$$

The false positive puzzle

- Test for a certain rare disease is correct 95% of times, and if the person does not have the disease, the results are negative with prob .95
- A random person drawn from the population has probability of .001 of having the disease
- Given that a person has just tested positive, what is the probability of having the disease?