Lecture outline

- Reading: Sections 1.5, 1.6
- Independence
  - Event couple
  - Multiple events
  - Conditional
  - Independent Trials and Binomial
- Counting
  - Principle
  - Permutations
  - Combinations
  - Partitions
Independence

• The occurrence of B does not change the probability of A, i.e., \( P(A|B)=P(A) \)
  \( \Rightarrow P(A \cap B)=P(A).P(B) \)

• Symmetric property

• Can we visualize this in terms of the sample space?

Independence?

• A and B disjoint \( \Rightarrow \) A and \( B^c \) are disjoint?

• \( P(A|B) = P(A|B^c) \)
Effect of conditioning on independence

- Assume that A and B are independent

![Diagram of events A, B, and C]

- If we are told that C occurred, are A and B independent?
- The events A and B are called conditionally independent if
  \[ P(A \cap B | C) = P(A | C)P(B | C) \]

Example

- Example: Rolling 2 dice, Let A and B are
  A={first roll is a 1}, B={second roll is a 4}
  C={The sum of the first and second rolls is 7}
- Is \( (A \cap B | C) \) independent?

![Diagram of dice rolls]
Independence vs. pairwise independence

• Another example: 2 independent fair coin tosses
  \( A = \{ \text{first toss is H} \} \), \( B = \{ \text{second toss is H} \} \)
  \( C = \{ \text{The two outcomes are different} \} \)

• \( P(C) = P(A) = P(B) \)
• \( P(C \cap A) = ? \)
• \( P(C | A \cap B) = ? \)

Independence of a collection of events

• Events \( A_1, A_2, \ldots, A_n \) are independent if

\[
P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)
\]

For every subset \( S \) of \( \{1, 2, \ldots, n\} \)

• For 3 events, this amounts to 4 conditions:
  – \( P(A_1 \cap A_2) \)
  – \( P(A_1 \cap A_3) \)
  – \( P(A_2 \cap A_3) \)
  – \( P(A_1 \cap A_2 \cap A_3) \)
Reliability example: network connectivity

- $P(\text{series subsystem succeeds})$
- $P(\text{parallel subsystem succeeds})$

Independent trials and the Binomial probabilities

- **Independent trials:** sequence of independent but identical stages
- **Bernoulli:** there are 2 possible results at each stage
Bernoulli trial

- $P(k) = P(k \text{ heads in an } n\text{-toss sequence})$
- From the previous page, the probability of any given sequence having $k$ heads: $p^k(1-p)^{n-k}$
- The total number of such sequences is $\binom{n}{k} p^k (1-p)^{n-k}$
- Where we define the notation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad k = 0, 1, \ldots, n$$

Review: Bernoulli trial

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$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad k = 0, 1, \ldots, n$$
- Probabilities add to 1

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$
Independence - summary

- Two events A and B are **independent** if
- \( P(A \cap B) = P(A) \cdot P(B) \) (Symmetric)
- If also, \( P(B) > 0 \), independence is equivalent to
- \( P(A | B) = P(A) \)
- A and B **conditionally independent** if given C, \( P(C) > 0 \), \( P(A \cap B | C) = P(A | C) \)
- Independence **does not imply** conditional independence and vice versa

Counting: Discrete Uniform Law

- Let all sample points be equally likely
  \[
P(A) = \frac{\text{number of elements in } A}{\text{number of points in the sample space}}
\]
- Counting is not always simple – challenging
- Parts of counting has to do with combinatorics
- We will learn some simple rules to help us count better
Counting principal

- $r$ steps
- At step $i$, there are $n_i$ choices
- Total number of choices $n_1, n_2, \ldots, n_r$

Two examples

- The number of telephone numbers
- The number of subsets of an $n$-element set

Permutations and combinations
k-Permutations

- The order of selection matters!
- Assume that we have n distinct objects
- **k-permutations**: find the ways that we can pick k out of these objects and arrange them in a certain order
  \[ n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!} \]
- **Permutations**: k=n case
- Example: Probability that 6 rolls of a six-sided die give different numbers

Combinations

- There is no certain order of the events
- **Combinations**: number of k-element subsets of a given n-element set
- Two ways of making an ordered sequence of k-distinct items:
  - Choose the k items one-at-a-time \(\frac{n!}{(n-k)!}\)
  - Choose k items then order them (k! possibilities) \(\binom{n}{k}\)
- Thus, \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\), \(k=0,1,\ldots,n\)
Different sampling methods

• Draw k balls from an Urn with n balls
  – Sampling with replacement and ordering
  – Sampling without replacement and ordering
  – Sampling with replacement without ordering
  – Sampling without replacement with ordering

Partitions

• Given an n-element set and nonnegative integers \( n_1, n_2, \ldots, n_r \) whose sum is n
• How many ways do we have to form the first subset? The second?...
• The total number of choices

\[
\binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2-\ldots-n_{r-1}}{n_r}
\]

• Several terms cancel: \( \frac{n!}{n_1! n_2! \cdots n_r!} \)
Examples

- The 52 cards are dealt to 4 players
- Find the probability that each one gets an ace?
- Count the size of the sample space
- Count the number of possible ways for Ace distribution
- Count the number of ways to Distribute the rest of 48 cards

Summary of counting results

- **Permutations** of n objects: $n!$
- **k-Permutations** of n objects: $n!/(n-k)!$
- **Combinations** of k out of n objects: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Partitions** of n objects into r groups, with the ith group having $n_i$ objects:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$