ELEC 303 – Random Signals

Lecture 5 – Probability Mass Function and Probability Density Function

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Lecture outline

• Reading: Section 2.1-2.3, and 3.1-3.3
• Discrete random variables
  – Probability mass functions (PMF)
• Continuous random variables
  – Probability density function (PDF)
Counting - summary

- **Permutations** of \( n \) objects: \( n! \)
- **k-Permutations** of \( n \) objects: \( n!/(n-k)! \)
- **Combinations** of \( k \) out of \( n \) objects:
- **Partitions** of \( n \) objects into \( r \) groups, with the \( i \)th group having \( n_i \) objects:

Random variables

- An assignment of a value (number) to every possible outcome
- It can be mathematically shown as a function from the sample space to the real numbers
  - Can be discrete or continuous
- Several random variables can be defined on the same sample space
- Our goal is to introduce some models applicable to many scenarios that involve random variables
Random variables

- A random variable is defined by a deterministic function that maps from the sample space to real numbers

\[ X(w) : \Omega \rightarrow R \]

Random variables - example

- The number of heads in a sequence of 5 coin tosses
- The 5-long sequence of H’s and T’s is not!
- The sum of values of two die rolls
- The time needed to transmit a message
- Temperature in Houston on Sept 9
- The number of words in your email
Motivation

Example
A coin is flipped three times. The sample space for this experiment is \( \Omega = \{ \text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} \} \).

Let random variable \( X \) be the number of heads in three coin tosses.

- \( X \) assigns each outcome in \( \Omega \) a number from the set \( \{0, 1, 2, 3\} \).

<table>
<thead>
<tr>
<th>HHH</th>
<th>HHT</th>
<th>HTH</th>
<th>HTT</th>
<th>THH</th>
<th>THT</th>
<th>TTH</th>
<th>TTT</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

There is nothing random about the Mapping!

Random Variables

- \( X \) maps \( w \in \Omega \) to the number \( X(w) \)
- The random variable is always denoted as \( X \), never as \( X(w) \)
- \( X(w) \) means the number assigned to the outcome \( w \), e.g. \( X(\text{HHH}) \) is 3 (nothing random)
- \( X \) is the random variable — one of its possible values is 3
- It is often convenient to not display the arguments of the functions when it is the functional relationship that is of interest:

\[
d(uv) = u \cdot dv + v \cdot du
\]

\[
y = h \cdot x
\]
Random variable (r.v.)

- A r.v. assigns a real number to each outcome of a random experiment
  - Flip a coin, \( X = 1 \) if heads and \( X = -1 \) if tails
  - **Discrete** r.v.: finite or countably infinite range
  - Measure the life of a device, \( X = \) the life
  - **Continuous** r.v.: the range of \( X \) contains an interval of real numbers
  - Denote random variables by \( X, Y \), and their values by \( x, y \)

Random Variables

- Two or more outcomes could have the same image but each outcome has exactly one image

\[
\begin{array}{cccccccc}
  \text{HHH} & \text{HHT} & \text{HTH} & \text{HHT} & \text{THH} & \text{THT} & \text{THH} & \text{TTT} \\
  3 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\
\end{array}
\]

- Consider the experiment consisting of tossing a coin till a Tail appears for the first time. \( X \) is the number of tosses on the trial
- \( X \) takes on values 1, 2, 3, 4, ...
Random variable (cont)

• **Example:** surface flaws in plastic panels used in the interior of automobiles are counted. Let $X = 1$ if the # of surface flaws ≤ 10
  
  2 if the # of surface flaws >10 and ≤ 20
  
  3 if the # of surface flaws >20 and ≤ 30
  
  4 if the # of surface flaws >30

  $X$ measures the level of quality

Random variable (cont)

• $X = 1$ if the # of surface flaws ≤ 10
  
  2 if the # of surface flaws >10 and ≤ 20
  
  3 if the # of surface flaws >20 and ≤ 30
  
  4 if the # of surface flaws >30

• What is the $P($# of surface flaws ≤ 10$)$?
• Alternatively what is $P(X=1)$?
Probability distribution

• Interested in
  \[ P(X=1)=f_1, \ P(X=2)=f_2, \ P(X=3)=f_3, \ P(X=4)=f_4 \]
• Probability distribution: a description of the probabilities associated with possible values of \( X \)
• For discrete r.v. \( X \) with possible values \( x_1, \ldots, x_n \), the probability mass function (pmf) is defined by \( f(x_i)=P(X=x_i) \).

\[
f(x_i) \geq 0, \quad \sum_{i=1}^{n} f(x_i) = 1
\]

Probability distribution (cont)

• Example: in a batch of 100 parts, 5 of them are defective. Two parts are randomly picked. Let \( X \) be the number of defective parts. What is the probability distribution of \( X \)?
Probability distribution (cont)

Example: in a batch of 100 parts, 5 of them are defective. Two parts are randomly picked. Let $X$ be the number of defective parts. What is the probability distribution of $X$?

- $P(X=0) = P(1^{st} \text{ is non-defective}, 2^{nd} \text{ is non-defective})$
  - $P(1^{st} \text{ is non-defective})P(2^{nd} \text{ is non-defective} | 1^{st} \text{ is non-defective})$ (here we use $P(A \cap B) = P(A)P(B | A)$)
  - $= (95/100)*(94/99) = 0.90202$

- $P(X=1) = P(1^{st} \text{ is defective, 2}^{nd} \text{ is non-defective}) + P(1^{st} \text{ is non-defective, 2}^{nd} \text{ is defective})$
  - $= (5/100)*(95/99) + (95/100)*(5/99) = 0.09596$

- $P(X=2) = P(1^{st} \text{ is defective, 2}^{nd} \text{ is defective})$
  - $= 5/100*4/99 = 0.00202$

Probability mass function

- Arrange probability mass function in a table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.90202</td>
</tr>
<tr>
<td>1</td>
<td>0.09596</td>
</tr>
<tr>
<td>2</td>
<td>0.00202</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$f(x_1)$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$f(x_2)$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$f(x_3)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Discrete random variables

• It is a real-valued function of the outcome of the experiments
  – can take a finite or infinitely finite number of values

• A discrete random variable has an associated probability mass function (PMF)
  – It gives the probability of each numerical value that the random variable can take

• A function of a discrete random variable defines another discrete random variable (RV)
  – Its PMF can be found from the PMF of the original RV

Probability mass function (PMF)

• Notations
  – Random variable: X
  – Experimental value: x
  – \( P_X(x) = P\{X=x\} \)

• It mathematically defines a probability law

• Probability axiom: \( \sum_x P_x(x) = 1 \)

• Example: Coin toss
  – Define \( X(H)=1 \), \( X(T)=0 \) (indicator RV)
Summary

• Probability mass function (PMF)
  \[ P_X(x) = P(X = x) \]
  \[ P(X \in S) = \sum_{x \in S} P_X(x) \]
  \[ P(X \in \Omega) = \sum_{x \in \Omega} P_X(x) = 1 \]

Computing PMF

• Collect all possible outcomes for which \( X = x \)
  \( \{ \omega \in \Omega, X(\omega) = x \} \)
• Add the probabilities
• Repeat for all \( x \)
• Example: Two independent tosses of a fair 6-sided die
  – \( F \): outcome of the first toss
  – \( S \): outcome of the second toss
  – \( Z = \min(F, S) \)
Continuous random variables

- Random variables with a continuous range of values
  - E.g., speedometer, people’s height, weight
- Possible to approximate with discrete
- Continuous models are useful
  - Fine-grain and more accurate
  - Continuous calculus tools
  - More insight from analysis

Probability density functions (PDFs)

- A RV is continuous if there is a non-negative PDF s.t. for every subset B of real numbers:
  \[ P(X \in B) = \int_B f_X(x) \, dx \]
- The probability that RV X falls in an interval is:
  \[ P(a \leq X \leq b) = \int_a^b f_X(x) \, dx \]
PDF (Cont’d)

- Continuous prob – area under the PDF graph
- For any single point: \( P(X = a) = \int_a^b f_X(x) \, dx = 0 \)
  \[ P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) \]
- The PDF function \((f_x)\) non-negative for every \(x\)
- Area under the PDF curve should sum up to 1
  \[ \int_{-\infty}^{+\infty} f_X(x) \, dx = P(-\infty < X < +\infty) = 1 \]
  \[ P(x \leq X \leq x + \delta) \approx f_X(x) \cdot \delta \]

PDF (example)

- A PDF can take arbitrary value, as long as it is summed to one over the interval, e.g.,
  \[ f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
  \[ \int_{-\infty}^{+\infty} f_X(x) \, dx = \int_0^1 \frac{1}{2\sqrt{x}} \, dx = \sqrt{x \bigg|_0^1} = 1 \]
Bernoulli random variable

- An event with two outcomes: Success & Fail
- Success (1) probability $-p$
- Fail (0) probability $1-p$

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

- State of a telephone: Free or busy
- State of machine: Working or broken down

Indicator random variable/Binomial

- Independently flip a coin $n$ times
- $X$: the number of heads in $n$ independent flips
- $P(H)=p$
- E.g., $n=3$
- Then, $P_X(2) = P(HHT)+P(HTH)+P(THH)=3p^2(1-p)$
- Generally speaking

$$p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$
Geometric random variable

- Coin toss
- $P(H) = p$
- $P(T) = 1 - p$
- $X$: # of tosses needed to observe the first head

$$p_x(x) = P(X = x) = p(1 - p)^{x-1}$$

Example

Q1. The MIT soccer team has 2 games scheduled for the weekend.
- $P$(not losing first game) $= 0.4$
- $P$(not losing second game) $= 0.7$

Results of the games are independent of each other. If it does not lose a game, it is equally to win or tie. The team receives 2 points for a win, 1 for a tie and 0 for a loss. Find the PMF of the number of points the team wins over the weekend.
Q1

$X =$ # of points the team earns on the weekend if $Y_i =$ points received from game $i$.

$X \in \{0, 1, 2, 3, 4, 5\}$  $Y_i \in \{0, 1, 2\}$

$P(X=0) = P(Y_i=0 | Y_{\neq i} = 0) = 0.6 \times 0.5 = 0.3$

$P(X=1) = P(Y_i=0 | Y_{\neq i} = 0) + P(Y_i=1 | Y_{\neq i} = 0) = 0.6 \times 0.5 + 0.4 \times 0.5 = 0.5$

$P(X=2) = P(Y_i=0 | Y_{\neq i} = 0) + P(Y_i=1 | Y_{\neq i} = 0) + P(Y_i=2 | Y_{\neq i} = 0)$

$= 0.6 \times 0.5 \times 0.5 + 0.4 \times 0.5 \times 0.5 + 0.2 \times 0.5 \times 0.5 = 0.54$

$P(X=3) = P(Y_i=1 | Y_{\neq i} = 0) + P(Y_i=2 | Y_{\neq i} = 0) = 0.4 \times 0.5 \times 0.5 \times 0.5 = 0.11$

$P(X=4) = P(Y_i=2 | Y_{\neq i} = 0) = 0.2 \times 0.5 \times 0.5 \times 0.5 = 0.02$

Example

Q2. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you?

...
Q2

\[ X \sim \text{Poisson} \left( \frac{364}{365} \right) \]

\[ P(X=1) = \left( \frac{364}{365} \right)^1 \cdot \left( \frac{1}{365} \right)^0 \approx 0.3486 \]

\[ X \text{ is the number of people that have the same birthday as me.} \]

\[ X \sim \text{Bernoulli} \left( \frac{364}{365} \right) \]