ELEC 303 – Random Signals

Lecture 6 – Random Variable, Mean, Variance

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Lecture outline

• Reading: Sections 2.2-2.4, 3.2-3.3
• Discrete random variables
• Continuous random variables
• Mean and variance
Review – random variable

• A random variable is defined by a deterministic function that maps from the sample space to real numbers

PMF (review)

• Probability mass function (PMF)

\[ P_X(x) = P(X = x) \]

\[ P(X \in S) = \sum_{x \in S} P_X(x) \]

\[ P(X \in \Omega) = \sum_{x \in \Omega} P_X(x) = 1 \]
Computing PMF

- Collect all possible outcomes for which $X=x$
  
  \[ \{ \omega \in \Omega, X(\omega) = x \} \]

- Add the probabilities

- Repeat for all $x$

- Example: Two independent tosses of a fair 6-sided die
  - $F$: outcome of the first toss
  - $S$: outcome of the second toss
  - $Z = \min(F,S)$

Bernoulli random variable

- An event with two outcomes: Success & Fail
- Success (1) probability $- p$
- Fail (0) probability $- 1 - p$

\[
p_x(x) = \begin{cases} 
  p & \text{if } x = 1 \\
  1 - p & \text{if } x = 0 
\end{cases}
\]

- State of a telephone: Free or busy
- State of machine: Working or broken down
Indicator random variable/Binomial

- Independently flip a coin n times
- \(X\): the number of heads in n independent flips
- \(P(H) = p\)
- E.g., \(n = 3\)
- Then, 
  \[P_X(2) = P(HHT) + P(HTH) + P(THH) = 3p^2(1-p)\]
- Generally speaking
  \[p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n\]

Geometric random variable

- Coin toss
- \(P(H) = p\)
- \(P(T) = 1 - p\)
- \(X\): # of tosses needed to observe the first head
  \[p_X(x) = P(X = x) = p(1 - p)^{x-1}\]
**Example**

**Q1.** The MIT soccer team has 2 games scheduled for the weekend.

- \( P(\text{not losing first game}) = 0.4 \)
- \( P(\text{not losing second game}) = 0.7 \)

Results of the games are independent of each other. If it does not lose a game, it is equally to win or tie. The team receives 2 points for a win, 1 for a tie and 0 for a loss. Find the PMF of the number of points the team wins over the weekend.

\[
X = \text{# of points the team earns over the weekend} \quad \text{PMF of } X
\]

\[
Y_1 = \text{points received from game 1} \quad i = 1, 2
\]

\[
X \in \{0, 1, 2, 3, 4\} \quad Y_i \in \{0, 1, 2\}
\]

\[
P(X=0) = P\{Y_1=0 \cap Y_2=0\} = 0.6 \times 0.3 = 0.18
\]

\[
P(X=1) = P\{y_1=0 \cap y_2=1\} + P\{y_1=1 \cap y_2=0\} = 0.6 \times 0.3 + 0.4 \times 0.7 = 0.32
\]

\[
P(X=2) = P\{y_1=0 \cap y_2=2\} + P\{y_1=1 \cap y_2=1\} + P\{y_1=2 \cap y_2=0\}
\]

\[
= 0.6 \times 0.3 \times 0.3 + 0.4 \times 0.7 \times 0.3 + 0.4 \times 0.7 \times 0.3 = 0.34
\]

\[
P(X=3) = P\{y_1=1 \cap y_2=2\} + P\{y_1=2 \cap y_2=1\} = (0.4 \times 0.7 \times 0.3) + (0.4 \times 0.7 \times 0.3) = 0.14
\]

\[
P(X=4) = P\{y_1=2 \cap y_2=2\} = 0.4 \times 0.7 \times 0.3 = 0.02
\]
Example

Q2. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you?

\[ P(X = 1) = \binom{499}{1} \left( \frac{1}{365} \right) \left( \frac{364}{365} \right)^{498} \approx 0.2486 \]

\( X \) is the number of people that have the same birthday as you. 
\( X \sim \text{Binomial}(n, p) \)

\[ X \sim \text{Binomial}(498, \frac{1}{365}) \]
PDF (review)

- A RV is continuous if there is a non-negative PDF s.t. for every subset $B$ of real numbers:
  $$P(X \in B) = \int_B f_X(x) \, dx$$
- The probability that RV $X$ falls in an interval is:
  $$P(a \leq X \leq b) = \int_a^b f_X(x) \, dx$$

Figure courtesy of Bertsekas&Tsitsiklis, Introduction to Probability, 2008

PDF (Cont’d)

- Continuous prob – area under the PDF graph
- For any single point: $P(X = a) = \int_a f_X(x) \, dx = 0$
- $P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$
- The PDF function ($f_X$) non-negative for every $x$
- Area under the PDF curve should sum up to 1
  $$\int_{-\infty}^{+\infty} f_X(x) \, dx = P(-\infty < X < +\infty) = 1$$
  $$P(x \leq X \leq x + \delta) \approx f_X(x) \cdot \delta$$
PDF (example)

- A PDF can take arbitrary value, as long as it is summed to one over the interval, e.g.,

\[ f_X(x) = \begin{cases} 
\frac{1}{2\sqrt{x}}, & \text{if } 0 \leq x \leq 1 \\
0, & \text{otherwise} 
\end{cases} \]

\[
\int_{-\infty}^{+\infty} f_X(x) \, dx = \int_{0}^{1} \frac{1}{2\sqrt{x}} \, dx = \sqrt{x}\bigg|_{0}^{1} = 1
\]

Mean

- Location (center) of a probability distribution

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.90202</td>
</tr>
<tr>
<td>1</td>
<td>0.09596</td>
</tr>
<tr>
<td>2</td>
<td>0.00202</td>
</tr>
</tbody>
</table>

- Wrong: \((0 + 1 + 2)/3 - 1\)

- Probability weighted average:

\(0 \times 0.90202 + 1 \times 0.09596 + 2 \times 0.00202 - 0.1\)
Mean (Cont’d)

• The mean of a discrete random variable $X$
\[ \mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \]

$f(x_i)$ is the probability mass function
$n$ here is the number of possible values for $X$
and it can be infinity

Mean (Cont’d)

• $X$ is a r.v. For any function $h$, $h(X)$ is still a r.v.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$h(x) = x^2$</th>
<th>$h(x) = 2x + 1$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.90202</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0.09596</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>0.00202</td>
</tr>
</tbody>
</table>

• Expected value of $h(X)$
\[ E(h(X)) = h(x_1) f(x_1) + \cdots + h(x_n) f(x_n) + \sum_{i=1}^{n} h(x_i) f(x_i) \]
• $E(X^2) = 0*0.90202 + 1*0.09596 + 4*0.00202 - 0.10404$
Mean (Expectation)

- Definition: $\mathbb{E}[X] = \sum x \cdot p_X(x)$
- Interpretations
  - Center of gravity for the PMF
  - Average in a large number of repetitions for one experiment

Example: a sample space $\Omega = \{0, 1, \ldots, n\}$ with a uniform probability law

$$E[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + \cdots + \frac{n}{n+1} = ?$$

Mean (Cont’d)

- Note that $E(X^2) - 0.10404 + (E(X))^2 - 0.01$
  - Generally, $E(h(X)) \neq h(E(X))$
- Expectation is linear:
  $$E(aX + b) = \sum_{i=1}^{n} (ax_i + b) f(x_i)$$
  $$= a \sum_{i=1}^{n} x_i f(x_i) + b \sum_{i=1}^{n} f(x_i) = aE(X) + b$$
Properties of expectation

• Let X be a RV and let Y=g(X)
  – It is often hard to calculate $E[Y]=\sum_y yP_Y(y)$
  – It is easier to compute: $E[Y]=\sum_x g(x)P_X(x)$

• Second moment: $E[X^2]$

• Generally speaking, $E[g(X)] \neq g(E[X])$

• Variance:
  \[
  \text{var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 \cdot p_X(x)
  \]

Properties of Expectation

• If $\alpha$ and $\beta$ are constants, and X and Y are RVs:
  – $E[\alpha]=$
  – $E[\alpha X]=$
  – $E[\alpha X+ \beta]=$
  – $E[X+Y]=$
  – $E[X.Y]=$
Variance

- Note that
  \[ E(X') = 0.10404 + (E(X))^2 - 0.01 \]
  - Generally, \( E(h(X)) \neq h(E(X)) \)
- Expectation is linear:
  \[ E(aX + b) = \sum_{i=1}^{n} (ax_i + b) f(x_i) \]
  \[ a \sum_{i=1}^{n} x_i f(x_i) + b \sum_{i=1}^{n} f(x_i) = aE(X) + b \]

Variance (Cont’d)

- Dispersion of a probability distribution
- Let \( X, Y \) be two random variables
- Have the same mean \( \mu_X - \mu_Y = 0 \)
- \( Y \) is much more scattered
- Probability weighted average of the squared distances to the mean

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f_X(x) )</th>
<th>( y )</th>
<th>( f_Y(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.25</td>
<td>-100</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>100</td>
<td>0.25</td>
</tr>
</tbody>
</table>

- For \( X \): \( 0.25*1^2 - 0.5*0^2 + 0.25*1^2 - 0.5 \)
- For \( Y \): \( 0.25*100^2 + 0.5*0^2 + 0.25*100^2 - 5000 \)
Variance (Cont’d)

- The variance of a discrete r.v. $X$
  \[ \sigma^2 = V(X) = E(X - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) \]

- Alternatively,
  \[ \sigma^2 = E(X^2) - (\mu^2) = \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2 \]

- Standard deviation $\sigma$ is the square root of variance: same unit.

Variance (Cont’d)

- Recall that $E(aX + b) = aE(X) + b$
- For a discrete r.v. $X$ with mean $\mu$, let $Y = aX + b$, where $a$ and $b$ are constants
  \[ V(Y) = E(Y - \mu_Y)^2 = E(aX + b - a\mu - b)^2 \]
  \[ = E((aX - a\mu)^2 = a^2 E(X - \mu)^2 = a^2 V(X) \]

  therefore, $V(aX+b) = a^2 V(X)$
- In particular, for a constant $b$, $V(b) = 0$
Discrete uniform distribution

- A r.v. has a discrete uniform distribution if each of the \( n \) values in its range, \( x_1, x_2, \ldots, x_n \), has equal probability \( 1/n \)
- **Example:** toss a fair die, let \( X \) be the outcome
  - Possible values of \( X \) are \{1,2,3,4,5,6\}
  - Each outcome has probability \( 1/6 \)
  - Mean \( E(X) = 3.5 \)

Average speed vs. average time

- If weather is good (probability=0.6) Alice walks the 2 miles with speed 5 miles/hr.
- If weather is bad, Alice rides her motorcycle at a speed \( V=30 \) miles/hr.

- What is the mean of the time \( T \) to get to the class?
- **Correct Solution** - derive the PMF of \( T \):
  - \( P_T(t) = 0.6 \), if \( t = 2/5 \); \( P_T(t) = 0.4 \), if \( t = 2/30 \)
  - \( E[T] = 0.6 \times 2/5 + 0.4 \times 2/30 = 4/15 \) hrs = 16 mins
Average speed vs. average time

- If weather is good (probability=0.6) Alice walks the 2 miles with speed 5 miles/hr.
- If weather is bad, Alice rides her motorcycle at a speed V=30 miles/hr.

What is the mean of the time $T$ to get to the class?
- **Mistake:** it is wrong to find the average speed $E[V] = 0.6 \times 5 + 0.4 \times 30 = 15$ miles/hr
  $\Rightarrow E[T] = 2/E[V] = 2/15$ hrs = 8 mins
- **Summary:** $E[T] = E[2/V] \neq 2/E[V]$

Mean and variance for continuous variables

- Expectation $E[X]$ and n-th moment $E[X^n]$ are defined similar to discrete
- A real-valued function $Y=g(X)$ of a continuous RV is a RV: $Y$ can be both continuous or discrete

- $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$
- $E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, dx$
- $\text{var}(X) = \sigma_X^2$
  $= \int_{-\infty}^{\infty} (x - E[X])^2 \cdot f_X(x) \, dx$
### Mean and variance of Uniform RV

- \( f_X(x) = \frac{1}{b-a} \quad a \leq x \leq b \)
- \( E[X] = \frac{a+b}{2} \)
- \( \sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx = \frac{(b-a)^2}{12} \)

### Exponential RV

\[
f_X(x) = \begin{cases} 
\lambda e^{-\lambda x}, & \text{if } x \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]

- \( \lambda \) is a positive RV characterizing the PDF
- E.g., time interval between two packet arrivals at a router, the lifetime of a bulb
- The probability that \( X \) exceeds a certain value decreases exponentially, for any \( (a \geq 0) \) we have

\[
P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x} dx = e^{-\lambda a}
\]

- Mean?
- Variance?