Practice Midterm 1 – Solutions

Q1.
First we can draw a tree with the the following branches:

Then, using the PDFs given in the question we can determine two probabilities that are clearly relevant for this question and give branch labels for the tree:

\[
P \left( Y \leq \frac{1}{4} \right) = \int_{0}^{\frac{1}{4}} f_{Y}(y) \, dy = \frac{1}{4}
\]

\[
P \left( W \leq \frac{1}{4} \right) = \int_{0}^{\frac{3}{4}} f_{W}(w) \, dw = \frac{3}{4}
\]

Finally, Bayes’s Rule gives

\[
P \left( A \mid X < \frac{1}{4} \right) = \frac{P(A)P(X \leq \frac{1}{4} \mid A)}{P(X \leq \frac{1}{4})} = \frac{P(A)P(Y \leq \frac{1}{4})}{P(X \leq \frac{1}{4})}
\]

\[
= \frac{P(A)P(Y \leq \frac{1}{4})}{P(A)P(X \leq \frac{1}{4}) + P(B)P(X \leq \frac{1}{4})}
\]

\[
= \frac{P(A)P(Y \leq \frac{1}{4})}{P(A)P(Y \leq \frac{1}{4}) + P(B)P(W \leq \frac{1}{4})}
\]

\[
= \frac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4}} = \frac{1}{4}.
\]

Q2.
(a) We know that the total length of the edge for red interval is two times that for black interval.
Since the ball is equally likely to fall in any position of the edge, probability of falling in a red interval is \(\frac{2}{3}\).

(b) Conditioned on the ball having fallen in a black interval, the ball is equally likely to fall anywhere in the interval. Thus, the PDF is

\[
f_{Z \mid \text{black interval}}(z) = \begin{cases} 
\frac{15}{\pi r}, & z \in \left[0, \frac{\pi r}{15}\right] \\
0, & \text{otherwise}
\end{cases}
\]
Q3.

\[ F_Z(z) = P(\min\{X, Y\} \leq z) \]
\[ = 1 - P(\min\{X, Y\} > z) \]
\[ = 1 - P(X > z, Y > z) \]
\[ = 1 - P(X > z)P(Y > z) \]
\[ = 1 - e^{-\lambda z}e^{-\mu z} \]
\[ = 1 - e^{-(\lambda + \mu)z}. \]

Z is an exponential random variable with parameter \( \lambda + \mu \), thus its pdf can be found.