ELEC 243
Problem Set 3
Homework Section
Due: February 6, 2015

H3.1 Express the following in both *Cartesian* (rectangular) form, $a + jb$, and *polar form*, $A\angle\Theta$, and plot in the complex plane ($j = \sqrt{-1}$; $*$ indicates complex conjugate).

- **(a)** $3\angle 60\,\mathrm{deg}$
- **(b)** $(4 + j5)^*$
- **(c)** $(2 + j4) \times (4 + j3)$
- **(d)** $-4 - j2$

H3.2 Fun with impulses:

- **(a)** Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{t-1} \cos\left[\frac{\pi}{2}(t-5)\right] \delta(t-3)dt$
- **(b)** Evaluate $\displaystyle\int_{-\infty}^{\infty} \delta(t-4)\sin(\pi t)dt$
- **(c)** Show that $\pi \cdot \delta(\sin \pi x) = \mathrm{III}(x) = \displaystyle\sum_{n=-\infty}^{\infty} \delta(x-n)$, the sampling function.

H3.3 For the signal $s(t) = \sin(2\pi f t + \theta)$:

- **(a)** Show that the phase, $\theta \neq 0$, has the effect of a time shift, $s(t-\tau)$.
- **(b)** If $f = 15\mathrm{Hz}$ what phase will delay the signal by 50 ms (shift it to the right)?
- **(c)** Express $s(t)$ as the Real part of a complex exponential.

H3.4 **Jean Baptiste Joseph Fourier** (21 March 1768 - 16 May 1830) was a French mathematician and physicist best known for developing the Fourier series and its application to problems of heat transfer. Fourier is also generally credited with the discovery of the greenhouse effect. He demonstrated that any periodic function can be decomposed into, or represented as, a linear combination of cosines, sines, or both. Given a periodic function of time: $v(t) = v(t + nT)$, where T is the period and n is an integer, the cosine representation is

$$v(t) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(2\pi n f_0 t + \theta_n)$$

Where $f_0 = \frac{1}{T}$ is the fundamental frequency. Each term is also periodic with period T (except the zero-frequency term a_0), and has two parameters, the real amplitude a_n and the phase θ_n. In many calculations it is easier to deal with complex exponentials than with sinusoids. Use Euler's identity to show that the expression above is identical to the complex exponential form:

$$v(t) = \sum_{n=-\infty}^{+\infty} c_n \exp(j2\pi n f_0 t)$$

Where the Fourier coefficients $c_n$are complex numbers: called the **spectrum** of the signal. You might find it helpful to review Section 4.13.1 and example 4.16 of the text.

H3.5 A system $y(t) = S[x(t)]$ can be represented by the differential equation $a\frac{d}{dt}y + by = x$.

- **(a)** Show that y may be computed as a function of x using the following formula:

$$y = \frac{1}{a}\int_{-\infty}^{t}(x(t') - by(t'))dt'$$

- **(b)** Show that the previous formula is equivalent to:

$$y = \frac{1}{a}\int_{0}^{t}(x(t') - by(t'))dt' + y(0)$$

- **(c)** Is this process physically realizable? (Assume that you have blocks available that can perform addition, subtraction, multiplication by a constant, integration, and differentiation.) If so, draw a block diagram of the system. If not, explain why.

H3.6 The goal of this problem is to show that the symmetry properties of a complex frequency transfer function (and by extension, a frequency spectrum) for a real system must be

$$|H(f)| = |H(-f)| \text{and} \angle H(f) = -\angle H(-f)$$

that is, the magnitude must be an even function and the angle must be an odd function of frequency.

Suppose that the input to a real LTI system is the real signal $\cos 2\pi gt$, and we want to find the output signal, which must also be a real signal of time, $v(t)$ in the form of a cosine. The frequency g is a positive real number, such as 100 Hz. All we know about the system (and all we need to know) is its complex frequency response $H(f)$ to complex exponential inputs, meaning we know the input ¯ output pair

$$e^{j2\pi ft} \to H(f)e^{j2\pi ft} = |H(f)|e^{j(2\pi ft + \theta(f))}$$

for all values of f, positive and negative, where $\theta(f)$ is the angle of $H(f)$.

- **(a)** Express the input signal $\cos 2\pi gt$ as a sum of scaled complex exponentials.
- **(b)** Using the frequency transfer function, find the output signal, which could, in general, be complex, since it is composed of complex terms.
- **(c)** Separate the output $v(t)$ into its real and imaginary parts using Euler's Rule.
- **(d)** Show that the above symmetry relationships must be true so that the imaginary part is zero. (Or at least if they are true the imaginary part is zero.)
- **(e)** Write an expression for the resulting real output signal $v(t)$.

Congratulations! You have made a safe trip from the time domain to the frequency domain and back. Note that in future you can solve part (e) just by taking the Real part of (b).

H3.7 A system $y(t) = S[x(t)]$ can be represented by the differential equation $\frac{d}{dt}y + by = x$. (This is the same system as in problem H3.4 with the equation normalized to make the arithmetic easier.)

- **(a)** Let $x(t) = \cos(\omega t)$. Using only real arithmetic, show that $y(t) = A\cos(\omega t + \phi)$. Find A and ϕ in terms of b and ω.
- **(b)** Let $x(t) = e^{j\omega t}$. Show that $y(t) = Ae^{j\omega t}$ (A will be complex). Find A in terms of b and ω.