## 10 Additive Gaussian Channel

The additive Gaussian channel, described by $Y = X + N$, with $N \sim \mathcal{N}(0, 1)$ and $X$ any random variable having a finite second moment, has very interesting properties. Here we derive another one.

**(a)** First of all, derive the following general result. Let $X$, $Y$ be arbitrary random variables and $Z$ an arbitrary auxiliary random variable. Show that

$$\mathcal{I}(X;Y) = \mathcal{E}_X\left[\mathcal{D}(p(Y|X)\|p(Z))\right] - \mathcal{D}(p(Y)\|p(Z))$$

where $\mathcal{E}_X[\cdot]$ means the expected value with respect to the random variable $X$.

**(b)** Let $X'$ denote a Gaussian random variable that models $X$. Here, "models" means $E[X'] = E[X]$ and $\sigma^2_{X'} = \sigma^2_X$. Let $Y'$ denote the output of the additive Gaussian channel when the Gaussian model serves as the input. Show that

$$\mathcal{D}(p(Y)\|p(Y')) = \mathcal{I}(X';Y') - \mathcal{I}(X;Y)\ .$$

**(c)** Conclude that the maximal statistical dependence between the input and output of the additive Gaussian channel occurs when the input is Gaussian.