3 Data Processing Theorem

Let the rate parameter of a Poisson random variable equal an exponential random variable.

$$P_{N|A}(n \mid A) = \frac{A^n e^{-A}}{n!}$$
$$p_A(a) = \alpha e^{-\alpha a} \mathrm{u}(a)$$

This model abstractly describes the situation where a Poisson counting process, which serves as the output, has its rate determined by an input, here an exponential random variable. Note that in this model, A is a continuous random variable and N is discrete. The Data Processing Theorem should apply regardless of the nature of the input and output. Let's confirm that.

(a) Determine the entropies of the input and and the output. Show that there is no general relation between the two (e.g, the entropy of the input can be greater or less than the entropy of the output).

(b) Assume the input distribution changes, remaining exponential but with different parameter values α_1, α_2. Find the Kullback-Leibler distance between these input probability functions. Confirm that this distance is positive.

(c) Find the Kullback-Leibler distance between the two outputs that result from the two input distributions.

(d) Determine the ratio γ of the output and input Kullback-Leibler distances. Show that is less than one for all choices of α_1, α_2.

(e) Does γ change if the Kullback-Leibler distances are calculated in "the other order?" In other words, compute γ for two cases: when $\mathcal{D}(p_A(a;\alpha_1) \| p_A(a;\alpha_2))$ defines the order of its arguments and when $\mathcal{D}(p_A(a;\alpha_2) \| p_A(a;\alpha_1))$ is used.