HOMEWORK 3 — Random Processes II

Exercise 1. Sampling Random Processes

A discrete-time stochastic process $Y_n \equiv X_{nT}$ is obtained by periodic sampling of a continuous-time zero-mean stationary process $X_t$ where $T$ is the sampling interval, i.e. $f_s = \frac{1}{T}$ is the sampling rate.

(a) Determine the relationship between the autocorrelation function of $X_t$ and the autocorrelation sequence of $Y_n$.

(b) Express the power density spectrum of $Y_n$ in terms of the power density spectrum of $X_t$.

(c) Determine the conditions under which the power density spectrum of $Y_n$ is equal to that of $X_t$.

Exercise 2. Processing Gaussian Noise

Suppose that white Gaussian noise $X_t$ is the input to a linear system with a transfer function defined by

$$H(f) = \begin{cases} 1 & |f| \leq 2 \\ 0 & |f| > 2 \end{cases}$$

Suppose further that the input process is zero mean and has spectral height $\frac{N_0}{2} = 5$. Let $Y_t$ denote the resulting output process.

(a) Find the power spectral density of $Y_t$. Find the autocorrelation of $Y_t$, $R_Y(\tau)$.

(b) Form a discrete time process by sampling $Y_t$ at time instants $T$ seconds apart, i.e.,

Find a value for $T$ such that these samples are uncorrelated. Are these samples also independent?

(c) What is the variance of each sample in the output process?
Exercise 3. Integrators

Suppose that $X_t$ is a zero mean white Gaussian process with spectral height $\frac{N_0}{2} = 5$. Denote $Y_t$ as the output of an integrator with input $X_t$.

\[ \int_0^t d\tau \]

(a) Find the mean function $\mu_Y(t)$. Find the autocorrelation function of $Y_t$, $R_Y(t + \tau, \tau)$.

(b) Is $Y_t$ a wide-sense stationary process?

(c) Let $Z_k$ be a sequence of random variables obtained by sampling $Y_t$ every $T$ seconds and dumping the samples, such that

\[ Z_k = \int_{(k-1)T}^{kT} X_\tau d\tau \]

Find the autocorrelation of the discrete-time process $Z_k$, $R_Z(k + m, k) = E[Z_{k+m}, Z_k]$.

(d) Is $Z_k$ a wide-sense stationary process?

Exercise 4. LTI Processing

$X_t$ is a stationary random process with autocorrelation function $R_X(\tau) = e^{-\alpha |\tau|}$, $\alpha > 0$. This process is applied to a LTI system with $h(t) = e^{-\beta t}u(t)$, where $\beta > 0$. Find the power spectral density of the output process $Y(t)$. Treat the cases $\alpha \neq \beta$ and $\alpha = \beta$ separately.

Exercise 5. BONUS – Noise Removal

Let $Y(t) = X(t) + N(t)$, where $X(t)$ and $N(t)$ are signal and noise processes. It is assumed that $X(t)$ and $N(t)$ are jointly stationary and have autocorrelation functions $R_X(\tau)$ and $R_N(\tau)$, and crosscorrelation function $R_{XN}(\tau)$.

We would like to recover the signal from $Y(t)$, by passing it through a LTI system with impulse response $h(t)$ and transfer function $H(f)$. The recovered signal, denoted $\hat{X}(t)$, should be as close to $X(t)$ as possible in the mean-squared sense.

(a) (+2 points) Find the crosscorrelation between $\hat{X}(t)$ and $X(t)$ in terms of $h(t), R_X(\tau), R_N(\tau)$ and $R_{XN}(\tau)$.

(b) (+2 points) Show that the LTI system that minimizes $E[(X(t) - \hat{X}(t))^2]$ has the transfer function

\[ H(f) = \frac{S_X(f) + S_{XN}(f)}{S_X(f) + S_N(f) + 2 \Re[S_{XN}(f)]} \]

(c) (+1 point) Now assume that $X(t)$ and $N(t)$ are independent and that $N(t)$ is a zero mean white Gaussian process with power-spectral density $\frac{N_0}{2}$. Find the optimal $H(f)$ under these conditions. What is the corresponding mean-squared error in this case?

All assignments are covered by the Honor Code. You may work together, but do not directly copy the solutions of other students.