HOMEWORK 4 — Signalling and Detection

Exercise 1. Likelihood Ratio Test

Consider the following system:

\[ b \xrightarrow{\text{Modulator}} X_{\tau} \xrightarrow[]{} Y_{\tau} \xrightarrow[]{} Z_t \xrightarrow[]{} Z_{uT} \xrightarrow[]{} \text{Threshold} \xrightarrow[]{\hat{b}} \]

Assume that \( N_{\tau} \) is a white Gaussian process with zero mean and spectral height \( \frac{N_o}{2} \). If \( b \) is "0" then \( X_{\tau} = A_{PT}(\tau) \) and if \( b \) is "1" then \( X_{\tau} = -A_{PT}(\tau) \) where

\[ p_{PT}(\tau) = \begin{cases} 1 & 0 \leq \tau \leq T \\ 0 & \text{otherwise} \end{cases} \]

Suppose \( \Pr[b=1] = \Pr[b=0] = \frac{1}{2} \).

(a) Find the probability density function \( Z_T \) when bit "0" is transmitted and also when bit "1" is transmitted. Refer to these densities as \( f_{Z_T|H_0}(z) \) and \( f_{Z_T|H_1}(z) \), where \( H_0 \) denotes the hypothesis that bit "0" is transmitted.

(b) Consider the ratio of the above two densities, i.e.,

\[ \Lambda(z) = \frac{f_{Z_T|H_0}(z)}{f_{Z_T|H_1}(z)} \]

and its natural log, \( \ln(\Lambda(z)) \). A reasonable scheme to decide which bit was actually transmitted is to compare \( \ln(\Lambda(z)) \) to a fixed threshold \( \gamma \). \( \Lambda(z) \) is often referred to as the likelihood function and \( \ln(\Lambda(z)) \) as the log likelihood function. Given threshold \( \gamma \) is used to decide \( \hat{b} = 0 \) when \( \ln(\Lambda(z)) \geq \gamma \) then find \( \Pr[\hat{b} \neq b] \) (note that we will say \( \hat{b} = 1 \) when \( \ln(\Lambda(z)) < \gamma \).

(c) Find a \( \gamma \) that minimizes \( \Pr[\hat{b} \neq b] \).
Exercise 2. Matched Filters

The received signal in a binary communication system that employs antipodal signals is

\[ r(t) = s(t) + n(t) \]

where \( s(t) \) is shown in the above figure, and \( n(t) \) is AWGN with power-spectral density \( \frac{N_0}{2} \) W/Hz.

(a) Sketch the impulse response of the filter matched to \( s(t) \).
(b) Sketch the output of the matched filter to input \( s(t) \).
(c) Determine the variance of the noise of the output of the matched filter at \( t = 3 \).
(d) Determine the probability of error as a function of \( A \) and \( N_0 \).

Exercise 3. Orthogonal Signals

Consider a set of \( M \) orthogonal signal waveforms \( s_m(t), 1 \leq m \leq M, 0 \leq t \leq T \), all of which have the same energy \( \varepsilon \). Define a new set of \( M \) waveforms as:

\[ s'_m(t) = s_m(t) - \frac{1}{M} \sum_{k=1}^{M} s_k(t), \quad 1 \leq m \leq M, \quad 0 \leq t \leq T. \]

Show that the \( M \) signal waveforms \( \{s'_m(t)\} \) have equal energy, given by:

\[ \varepsilon' = \frac{(M-1)\varepsilon}{M} \]

and are equally correlated, with correlation coefficient:

\[ \gamma_{mn} = \frac{1}{\varepsilon'} \int_{0}^{\infty} s'_m(t)s'_n(t)dt = -\frac{1}{M-1} \]
Exercise 4. Correlated Noise

Suppose that two signal waveforms $s_1(t)$ and $s_2(t)$ are orthogonal over the interval $(0,T)$. A sample function $n(t)$ of a zero-mean white noise process is cross-correlated with $s_1(t)$ and $s_2(t)$ to yield:

$$n_1 = \int_0^T s_1(t)n(t)dt$$

$$n_2 = \int_0^T s_2(t)n(t)dt$$

Prove that $E[n_1n_2] = 0$.

Exercise 5. Decision Regions

Three equally probable messages $m_1$, $m_2$ and $m_3$ are to be transmitted over an AWGN channel with noise power spectral density $\frac{N_0}{2}$. The messages are:

$$s_1(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = -s_3(t)$$

$$\begin{cases} 1 & 0 \leq t \leq \frac{T}{2} \\ -1 & \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the dimensionality of the signal space?

(b) Find an appropriate basis for the signal space (Hint: You can find the basis by inspection without using the Gram-Schmidt procedure)

(c) Sketch the optimal decision region $R_1$, $R_2$ and $R_3$.

(d) Which of the three messages is more vulnerable to errors and explain why.

All assignments are covered by the Honor Code. You may work together, but do not directly copy the solutions of other students.