Sending Gaussian Sources over Gaussian Channels: Variations on a Theme by Goblick

Amos Lapidoth
Information and Signal Processing Laboratory
Swiss Federal Institute of Technology (ETH)
Zurich, Switzerland
lapidoth@isi.ee.ethz.ch

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Main Themes

• A continuum of optimal schemes for sending a Gaussian source over the Gaussian channel.

• Shannon’s source-channel separation approach and Goblick’s uncoded approach are but two extreme points.

• Below an SNR threshold, uncoded transmission is optimal for sending a bi-variate Gaussian over a Gaussian MAC.
The Single-User Set-Up

- **Source:**
  \[ \{ S_k \} \sim \text{iid} \mathcal{N}(0, \sigma^2) \]

- **Distortion**
  \[ d(s, \hat{s}) = (s - \hat{s})^2 \]

- **Channel**
  \[ Y_k = x_k + Z_k \]

- **Noise**
  \[ \{ Z_k \} \sim \text{iid} \mathcal{N}(0, N) \]

- **Encoder**
  \[ f_n : \mathbb{R}^n \to \mathbb{R}^n \]
  \[ s \mapsto (x_1(s), \ldots, x_n(s)) \]
The Single-User Set-Up Contd.

- **Constraint**
  \[ \frac{1}{n} \mathbb{E} \left[ \| f_n(S) \|^2 \right] \leq P \]

- **Reconstructor**
  \[ \phi_n : \mathbb{R}^n \to \mathbb{R}^n \]
  \[ \mathbf{y} \mapsto (\hat{s}_1(\mathbf{y}), \ldots, \hat{s}_n(\mathbf{y})) \]

- **Performance**
  \[ d(f_n, \phi_n) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ (S_k - \hat{S}_k)^2 \right] . \]
Some Shannon Theory

- Distortion-Rate function for a Gaussian source
  \[ D(R) = \sigma^2 2^{-2R} \]

- Channel Capacity
  \[ C = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right) \]

- The fundamental limit:
  \[ d(f_n, \phi_n) \geq D(R) \bigg|_{R=C} \]
  \[ = \sigma^2 2^{-2C} \bigg|_{R=\frac{1}{2} \log (1+\frac{P}{N})} \]
  \[ = \sigma^2 \frac{N}{P+N} \]
  \[ \triangleq D^* \]
We say that \( \{ f_n, \phi_n \} \) is asymptotically optimal if

\[
\lim_{n \to \infty} d(f_n, \phi_n) = D^*
\]
Source-Channel Separation

- Describe $s$ using $nR(D^*)$ bits.
- Send these $nR(D^*)$ bits using the channel $n$ times with a good blocklength-$n$ codebook of rate $C - \epsilon$.
- Decode bits.
- Reconstruct $s$. 
Uncoded Transmission

• Just scale source symbols:

\[ X_k = \sqrt{\frac{P}{\sigma^2}} S_k. \]

• Channel output

\[ Y_k = \sqrt{\frac{P}{\sigma^2}} S_k + Z_k \]

or

\[ \frac{Y_k}{\sqrt{P/\sigma^2}} = S_k + \mathcal{N} \left( 0, \frac{N\sigma^2}{P} \right). \]

• Reconstruction

\[ \hat{S}_k = \frac{\sigma^2}{\sigma^2 + N\sigma^2/P} \cdot \frac{Y_k}{\sqrt{P/\sigma^2}}. \]
Linear MMSE Refresher

If

\[ Y = S + W \]

where

\[ S \sim \mathcal{N}(0, \sigma^2) \]

and

\[ W \sim \mathcal{N}(0, \eta^2) \]

with \( W \) and \( S \) being independent, then

\[ \hat{S} = \frac{\sigma^2}{\sigma^2 + \eta^2} \cdot Y \]

and

\[ \mathbb{E} \left[ (S - \hat{S})^2 \right] = \sigma^2 \frac{\eta^2}{\sigma^2 + \eta^2} \]
Uncoded Transmission contd.

\[ \frac{Y_k}{\sqrt{P/\sigma^2}} = S_k + \mathcal{N}\left(0, \frac{N\sigma^2}{P}\right) \]

so that the MMSE performance is

\[ E\left[ (S_k - \hat{S}_k)^2 \right] = \sigma^2 \frac{\eta^2}{\sigma^2 + \eta^2} \]
\[ = \sigma^2 \frac{N\sigma^2/P}{\sigma^2 + N\sigma^2/P} \]
\[ = \sigma^2 \frac{N}{P + N} \]
\[ = D^* \]
A Continuum of Optimal Schemes

Here

\[ 0 < \rho < \frac{1}{2} \log \left(1 + \frac{P}{N}\right) \]

is arbitrary.

- **Uncoded** \((\rho = 0)\)
- **Source-Channel Separation** \((\rho = 1/2 \cdot \log(1 + P/N))\).
Producing the Channel Input

Quantizer:

- $2^{n\rho}$ codewords
- drawn independently
- uniformly over a centered sphere in $\mathbb{R}^n$

$$\frac{1}{n}\|u\|^2 \approx \sigma^2 - \Delta$$
$$\approx \sigma^2 - \sigma^2 2^{-2\rho}$$
$$= D(\rho).$$

VQ Output: With high probability

$$\exists u^* \in \mathcal{C} \text{ s.t. } \frac{\langle s - u^*, u^* \rangle}{\|s - u^*\| \cdot \|u^*\|} \approx 0$$

which implies

$$\frac{1}{n}\|s - u^*\|^2 \approx \sigma^2 2^{-2\rho}.$$ 

Set

$$u^* = \arg\min_{u \in \mathcal{C}} |\langle s - u, u \rangle|.$$
Producing the Channel Input Contd.

Channel input is a linear combination of VQ-output and source sequence:

\[ x = \alpha s + \beta u^* \]

\[ = (\alpha + \beta)u^* + \alpha(s - u^*) \]

where

\[ \beta(\rho) = \sqrt{\frac{P + N}{\sigma^2}} - \alpha(\rho), \]

\[ \alpha(\rho) = \sqrt{\frac{2^{-2\rho}(N + P) - N}{\sigma^22^{-2\rho}}}. \]  \hspace{1cm} (1)

To satisfy power constraint,

\[ (\alpha + \beta)^2\|u^*\|^2 + \alpha^2\|s - u^*\|^2 \approx nP \]

i.e.,

\[ (\alpha + \beta)^2\sigma^2(1 - 2^{-2\rho}) + \alpha^2\sigma^22^{-2\rho} \approx P. \]  \hspace{1cm} (3)
Two step reconstruction

Channel output

\[ Y = (\alpha + \beta)u^* + \alpha(s - u^*) + Z \]

\text{treat as noise}

Step 1: Decode \( u^* \) treating the quantization noise and the channel noise as white Gaussian noise.

\[ \hat{u}^* = \arg\max_{u \in \mathcal{C}} \langle y, u \rangle. \]

This will succeed with high probability if

\[ \rho < \frac{1}{2} \log \left( 1 + \frac{(\alpha + \beta)^2 \sigma^2 (1 - 2^{-2\rho})}{\alpha^2 \sigma^2 2^{-2\rho} + N} \right) \]

Setting this to hold with equality gives us the second equation in \( \alpha, \beta \).
Step 2 Assume $\hat{u}^* = u^*$ so that

$$Y - (\alpha + \beta)\hat{u}^* = \alpha(s - u^*) + Z$$

and estimate quantization noise:

$$\hat{s} = \hat{u}^* + w$$

where

$$w = \frac{\sigma^2 2^{-2\rho}}{\sigma^{2-2\rho} + N/\alpha^2} \cdot \frac{Y - (\alpha + \beta)\hat{u}^*}{\alpha}.$$
Correlated Sources over a Gaussian MAC

- **Source:**
  \[
  \{(S_{1,k}, S_{2,k})\} \sim \text{IID } \mathcal{N}(0, K_{SS})
  \]
  where
  \[
  K_{SS} = \begin{pmatrix}
  \sigma_1^2 & \rho \sigma_1 \sigma_2 \\
  \rho \sigma_1 \sigma_2 & \sigma_2^2
  \end{pmatrix}
  \]

- **Encoder 1:** maps \((S_{1,1}, \ldots, S_{1,n})\) to \(X_{1,1}, \ldots X_{1,n}\) using
  \[
f_1 : \mathbb{R}^n \to \mathbb{R}^n.
  \]

- **Encoder 2:** maps \((S_{2,1}, \ldots, S_{2,n})\) to \(X_{2,1}, \ldots X_{2,n}\) using
  \[
f_2 : \mathbb{R}^n \to \mathbb{R}^n.
  \]

- **Power Constraints:**
  \[
  \frac{1}{n} \mathbb{E} \left[ \left\| f_\nu \left( \{S_{\nu,k}\}_{k=1}^n \right) \right\|^2 \right] \leq P_\nu, \quad \nu = 1, 2.
  \]
The Set-Up contd.

- The Channel:
  \[ Y_k = x_{1,k} + x_{2,k} + Z_k \]
  where
  \[ \{Z_k\}_{k=1}^n \sim \text{IID } \mathcal{N}(0, N). \]

- Reconstructions:
  \[ \phi_1 : (Y_1, \ldots, Y_n) \mapsto (\hat{S}_{1,1}, \ldots, \hat{S}_{1,n}) \]
  \[ \phi_2 : (Y_1, \ldots, Y_n) \mapsto (\hat{S}_{2,1}, \ldots, \hat{S}_{2,n}) \]

Which pairs \((D_1, D_2)\) are achievable?
Some Remarks

1. Region depends on $|\rho|$. The sign of $\rho$ is immaterial.

2. Distortion scales linearly with $(\sigma_1^2, \sigma_2^2)$.

3. Region is convex.

4. We shall thus assume
   $$\sigma_1^2 = \sigma_2^2 \triangleq \sigma^2 \quad \rho > 0.$$  

We focus on the SYMMETRIC CASE where
   $$P_1 = P_2 \triangleq P.$$  
Achievability Results

- Uncoded Transmission
- Independent Gaussian Codebooks
- Time Sharing and Power Splitting $\rightarrow$ lower convex envelope
- Superposition of coded and uncoded transmission
Uncoded Transmission

\[ X_{1,k} = \sqrt{\frac{P}{\sigma^2}} S_{1,k}, \quad k = 1, \ldots, n. \]

\[ X_{2,k} = \sqrt{\frac{P}{\sigma^2}} S_{2,k}, \quad k = 1, \ldots, n. \]

\[ Y_k = \sqrt{\frac{P}{\sigma^2}} S_{1,k} + \sqrt{\frac{P}{\sigma^2}} S_{2,k} + Z_k. \]

Use a linear estimator for \( \hat{S}_{1,k} \) and \( \hat{S}_{2,k} \):

\[ D^*(\sigma^2, \rho, P, N) \leq \sigma^2 P(1 - \rho^2) + N \frac{2P(1 + \rho) + N}{2P(1 + \rho) + N} \]

Excellent for \( \rho = 1 \) but otherwise doesn’t tend to zero as \( P/N \to \infty \).
A Lower Bound on $D^*$

For a memoryless bi-variate source and $d_1(s_1, \hat{s}_1), d_2(s_2, \hat{s}_2) \geq 0$ the pair $(D_1, D_2)$ is achievable with powers $P_1, P_2$ only if

$$\min_{P_{\hat{s}_1, \hat{s}_2}|S_1, S_2} I(S_1, S_2; \hat{S}_1, \hat{S}_2)$$

such that

$$\mathbb{E}\left[(S_1 - \hat{S}_1)^2\right] \leq D_1,$$

$$\mathbb{E}\left[(S_2 - \hat{S}_2)^2\right] \leq D_2,$$

does not exceed

$$\frac{1}{2} \log \left(1 + \frac{P_1 + P_2 + 2\rho_{\max}\sqrt{P_1P_2}}{N}\right),$$

where $\rho_{\max} = \rho_{\max}(S_1, S_2)$ is the Hirschfeld-Gebelein-Rényi maximal correlation:

$$\rho_{\max} = \sup \mathbb{E}[g(S_1)h(S_2)]$$

where the supremum is over all functions $f, g$ under which

$$\mathbb{E}[g(S_1)] = \mathbb{E}[h(S_2)] = 0 \quad \mathbb{E}[g^2(S_1)] = \mathbb{E}[h^2(S_2)] = 1$$

(4)
In the Symmetric Gaussian Case

For $P_1 = P_2 = P$ we obtain

$$D^*(\sigma^2, \rho, P, N) \geq \begin{cases} \sigma^2 \frac{P(1-\rho^2) + N}{2P(1+\rho) + N} & \text{for } \frac{P}{N} \in \left(0, \frac{\rho}{1-\rho^2}\right] \\ \sigma^2 \sqrt{\frac{(1-\rho^2)N}{2P(1+\rho) + N}} & \text{for } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases}$$
On the Optimality of Uncoded Transmission

For

\[ \frac{P}{N} < \frac{\rho}{1 - \rho^2} \]

the bounds agree!

\[ D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1 - \rho^2) + N}{2P(1 + \rho) + N}, \quad \text{if} \quad \frac{P}{N} < \frac{\rho}{1 - \rho^2} \]

For \( P/N < \rho/(1 - \rho^2) \) uncoded transmission is optimal!
Idea Behind the Proof

• Use the Hirschfeld-Gebelein-Rényi maximal correlation to upper bound

\[ I(X_1 + X_2; Y). \]

• Use the Data Processing Inequality to use this upper bound to obtain an upper bound on

\[ I(S_1, S_2; \hat{S}_1, \hat{S}_2). \]

• Use this upper bound and the distortion-rate function to obtain a necessary condition on

\[ (D_1, D_2). \]
Deriving an upper bound on \( I(X_1 + X_2; Y) \)

- By simple algebra

\[
\text{Var}[X_{1,k} + X_{2,k}] = \text{Var}[X_{1,k}] + \text{Var}[X_{2,k}] + 2\rho(X_{1,k}, X_{2,k})\sqrt{\text{Var}[X_{1,k}] \cdot \text{Var}[X_{2,k}]}
\]

\[
\leq \text{Var}[X_{1,k}] + \text{Var}[X_{2,k}] + 2\tilde{\rho}_{\text{max}}\sqrt{\text{Var}[X_{1,k}] \cdot \text{Var}[X_{2,k}]}
\]

where \( \tilde{\rho}_{\text{max}} \) is the maximum correlation between any functional of \((S_{1,1}, \ldots, S_{1,n})\) and any functional of \((S_{2,1}, \ldots, S_{2,n})\).

- Our bi-variate source is IID so \( \tilde{\rho}_{\text{max}} \) is the maximum correlation between any two functionals of \( S_{1,1} \) and \( S_{2,1} \) (Witsenhausen’75)

\[
\text{Var}[X_{1,k} + X_{2,k}] \leq \text{Var}[X_{1,k}] + \text{Var}[X_{2,k}] + 2\rho_{\text{max}}\sqrt{\text{Var}[X_{1,k}] \cdot \text{Var}[X_{2,k}]}
\]

- Summing the variances we obtain using the Cauchy-Schwarz inequality

\[
\frac{1}{n} \sum_{k=1}^{n} \text{Var}[X_{1,k} + X_{2,k}] \leq P_1 + P_2 + 2\rho_{\text{max}}\sqrt{P_1P_2}.
\]
• But IID Gaussians maximize differential entropy subject to sum of variances, so

\[
\frac{1}{n} h(Y) \leq \frac{1}{2} \log \left( 2\pi e \left( P_1 + P_2 + 2\rho_{\text{max}} \sqrt{P_1 P_2} \right) \right)
\]

• and hence

\[
\frac{1}{n} I(X_1 + X_2; Y) \leq \frac{1}{2} \left( 1 + \frac{P_1 + P_2 + 2\rho_{\text{max}} \sqrt{P_1 P_2}}{N} \right)
\]
Some Bounds for $\rho = 0.4$

A. Lapidoth and S. Tinguely, “Sending a Bi-Variate Gaussian Source over a Gaussian MAC,” to be presented at ISIT’06.
Some High SNR Asymptotics

\[ D^* \approx \sigma^2 \cdot \sqrt{\frac{N}{P}} \cdot \sqrt{\frac{1 - \rho}{2}} \]

in the sense that the ratio of the two sides tends to one.