Sending Gaussian Sources over Gaussian Channels: Variations on a Theme by Goblick

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Main Themes

- A continuum of optimal schemes for sending a Gaussian source over the Gaussian channel.
- Shannon's source-channel separation approach and Goblick's uncoded approach are but two extreme points.
- Below an SNR threshold, uncoded transmission is optimal for sending a bi-variate Gaussian over a Gaussian MAC.

The Single-User Set-Up

- Source:

$$\{S_k\} \sim \text{ IID } \mathcal{N}\big(0, \sigma^2\big)$$

- Distortion

$$d(s, \hat{s}) = (s - \hat{s})^2$$

- Channel

$$Y_k = x_k + Z_k$$

- Noise

$$\{Z_k\} \sim \text{IID} \quad \mathcal{N}(0, N)\,.$$

- Encoder

$$f_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$$
$$\mathbf{s} \mapsto \big(x_1(\mathbf{s}), \ldots, x_n(\mathbf{s})\big).$$

The Single-User Set-Up Contd.

- Constraint

$$\frac{1}{n}\mathsf{E}\left[\left\|f_n(\mathbf{S})\right\|^2\right] \leq P$$

- Reconstructor

$$\phi_n : \mathbb{R}^n \to \mathbb{R}^n$$
$$\mathbf{y} \mapsto \big(\hat{s}_1(\mathbf{y}), \ldots, \hat{s}_n(\mathbf{y})\big)$$

- Performance

$$d(f_n, \phi_n) = \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_k - \hat{S}_k)^2\right].$$

Some Shannon Theory

- Distortion-Rate function for a Gaussian source

$$D(R) = \sigma^2 2^{-2R}$$

- Channel Capacity

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- The fundamental limit:

$$\begin{aligned} d(f_n, \phi_n) &\geq D(R)\Big|_{R=C} \\ &= \sigma^2 2^{-2R}\Big|_{R=\frac{1}{2}\log\left(1+\frac{P}{N}\right)} \\ &= \sigma^2 \frac{N}{P+N} \\ &\triangleq D^* \end{aligned}$$

(Asymptotically) Optimal Schemes

We say that $\{f_n, \phi_n\}$ is asymptotically optimal if

$$\lim_{n\to\infty} d(f_n, \phi_n) = D^*$$

Source-Channel Separation

- Describe $\mathbf{s}$ using $nR(D^*)$ bits.
- Send these $nR(D^*)$ bits using the channel n times with a good blocklength-n codebook of rate $C - \epsilon$.
- Decode bits.
- Reconstruct $\mathbf{s}$.

Uncoded Transmission

- Just scale source symbols:

$$X_k = \sqrt{\frac{P}{\sigma^2}} S_k.$$

- Channel output

$$Y_k = \sqrt{\frac{P}{\sigma^2}} S_k + Z_k$$

or

$$\frac{Y_k}{\sqrt{P/\sigma^2}} = S_k + \mathcal{N}\left(0, \frac{N\sigma^2}{P}\right).$$

- Reconstruction

$$\hat{S}_k = \frac{\sigma^2}{\sigma^2 + N\sigma^2/P} \cdot \frac{Y_k}{\sqrt{P/\sigma^2}}.$$

Linear MMSE Refresher

If

$$Y = S + W$$

where

$$S \sim \mathcal{N}\left(0, \sigma^2\right)$$

and

$$W \sim \mathcal{N}\left(0, \eta^2\right)$$

with W and S being independent, then

$$\hat{S} = \frac{\sigma^2}{\sigma^2 + \eta^2} \cdot Y$$

and

$$\mathsf{E}\left[(S - \hat{S})^2\right] = \sigma^2 \frac{\eta^2}{\sigma^2 + \eta^2}$$

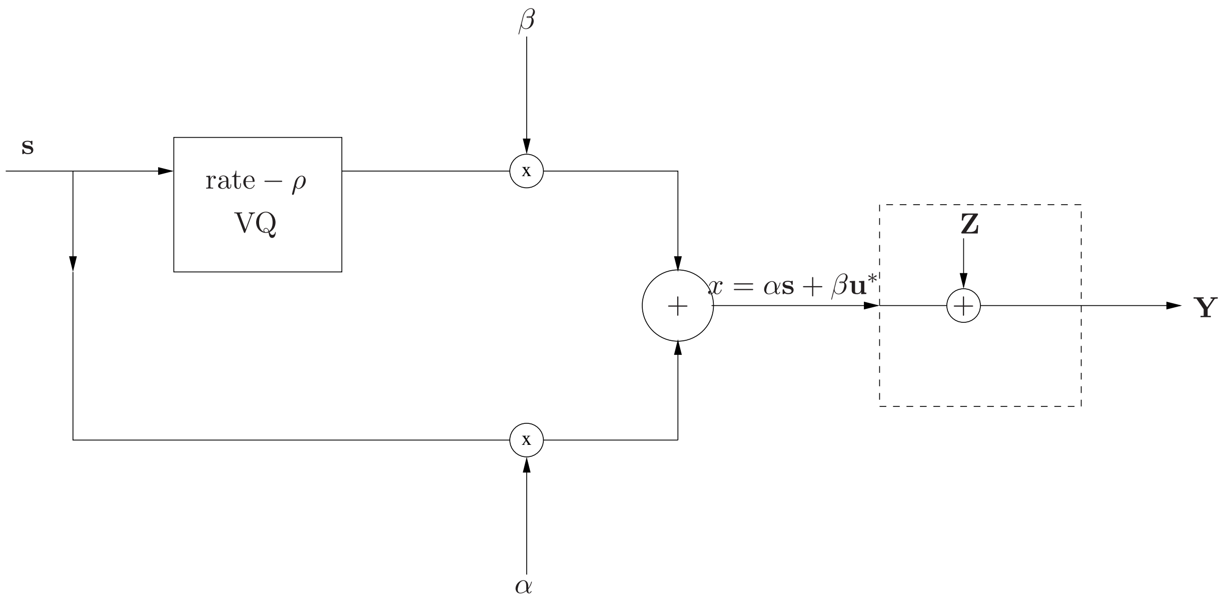
Uncoded Transmission contd.

$$\frac{Y_k}{\sqrt{P/\sigma^2}} = S_k + \mathcal{N}\left(0, \frac{N\sigma^2}{P}\right)$$

so that the MMSE performance is

$$\begin{aligned}
\mathsf{E}\left[(S_k - \hat{S}_k)^2\right] &= \sigma^2 \frac{\eta^2}{\sigma^2 + \eta^2} \\
&= \sigma^2 \frac{N\sigma^2/P}{\sigma^2 + N\sigma^2/P} \\
&= \sigma^2 \frac{N}{P + N} \\
&= D^*
\end{aligned}$$

A Continuum of Optimal Schemes



Here

$$0 < \rho < \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

is arbitrary.

- Uncoded $(\rho = 0)$
- Source-Channel Separation $(\rho = 1/2 \cdot \log(1 + P/N))$.

Producing the Channel Input

Quantizer :

– $2^{n\rho}$ codewords

– drawn independently

– uniformly over a centered sphere in $\mathbb{R}^n$

$$\begin{aligned}\frac{1}{n}\|\mathbf{u}\|^2 &\approx \sigma^2 - \Delta \\ &\approx \sigma^2 - \sigma^2 2^{-2\rho} \\ &= D(\rho).\end{aligned}$$

VQ Output : With high probability

$$\exists \mathbf{u}^* \in \mathcal{C} \text{ s.t. } \frac{\langle \mathbf{s} - \mathbf{u}^*, \mathbf{u}^* \rangle}{\|\mathbf{s} - \mathbf{u}^*\| \cdot \|\mathbf{u}^*\|} \approx 0$$

which implies

$$\frac{1}{n}\|\mathbf{s} - \mathbf{u}^*\|^2 \approx \sigma^2 2^{-2\rho}.$$

Set

$$\mathbf{u}^* = \operatorname{argmin}_{\mathbf{u} \in \mathcal{C}} |\, \langle \mathbf{s} - \mathbf{u}, \mathbf{u} \rangle \,|.$$

Producing the Channel Input Contd.

Channel input is a linear combination of VQ-output and source sequence:

$$\begin{aligned}\mathbf{x} &= \alpha\mathbf{s} + \beta\mathbf{u}^* \\ &= (\alpha+\beta)\mathbf{u}^* + \alpha(\mathbf{s}-\mathbf{u}^*)\end{aligned}$$

where

$$\beta(\rho) = \sqrt{\frac{P+N}{\sigma^2}} - \alpha(\rho), \qquad (1)$$

$$\alpha(\rho) = \sqrt{\frac{2^{-2\rho}(N+P)-N}{\sigma^2 2^{-2\rho}}}. \qquad (2)$$

To satisfy power constraint,

$$(\alpha+\beta)^2\|\mathbf{u}^*\|^2 + \alpha^2\|\mathbf{s}-\mathbf{u}^*\|^2 \approx nP$$

i.e.,

$$(\alpha+\beta)^2\sigma^2(1-2^{-2\rho}) + \alpha^2\sigma^2 2^{-2\rho} \approx P. \qquad (3)$$

Two step reconstruction

Channel output

$$\mathbf{Y} = (\alpha + \beta)\mathbf{u}^* + \underbrace{\alpha(\mathbf{s} - \mathbf{u}^*) + \mathbf{Z}}_{\text{treat as noise}}$$

Step 1: Decode $\mathbf{u}^*$ treating the quantization noise and the channel noise as white Gaussian noise.

$$\hat{\mathbf{u}}^* = \text{argmax}_{\mathbf{u} \in \mathcal{C}} \langle \mathbf{y}, \mathbf{u} \rangle .$$

This will succeed with high probability if

$$\rho < \frac{1}{2} \log \left(1 + \frac{(\alpha + \beta)^2 \sigma^2 (1 - 2^{-2\rho})}{\alpha^2 \sigma^2 2^{-2\rho} + N} \right)$$

Setting this to hold with equality gives us the second equation in α, β.

The Second Step in the Reconstruction

Step 2 Assume $\hat{\mathbf{u}}^* = \mathbf{u}^*$ so that

$$\mathbf{Y} - (\alpha + \beta)\hat{\mathbf{u}}^* = \alpha(\mathbf{s} - \mathbf{u}^*) + \mathbf{Z}$$

and estimate quantization noise:

$$\hat{\mathbf{s}} = \hat{\mathbf{u}}^* + \mathbf{w}$$

where

$$\mathbf{w} = \frac{\sigma^2 2^{-2\rho}}{\sigma 2^{-2\rho} + N/\alpha^2} \cdot \frac{\mathbf{Y} - (\alpha + \beta)\hat{\mathbf{u}}^*}{\alpha}.$$

Correlated Sources over a Gaussian MAC

- **Source:**

$$\{(S_{1,k}, S_{2,k})\} \sim \text{ IID } \mathcal{N}(0, K_{SS})$$

where

$$K_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

- **Encoder 1:** maps $(S_{1,1}, \ldots, S_{1,n})$ to $X_{1,1}, \ldots X_{1,n}$ using

$$f_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

- **Encoder 2:** maps $(S_{2,1}, \ldots, S_{2,n})$ to $X_{2,1}, \ldots X_{2,n}$ using

$$f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

- **Power Constraints:**

$$\frac{1}{n}\mathsf{E}\left[\| f_\nu\left(\{S_{\nu,k}\}_{k=1}^n\right)\|^2\right] \leq P_\nu, \quad \nu = 1, 2.$$

The Set-Up contd.

- **The Channel:**

$$Y_k = x_{1,k} + x_{2,k} + Z_k$$

where

$$\{Z_k\}_{k=1}^n \sim \text{ IID } \mathcal{N}(0, N) \,.$$

- **Reconstructions:**

$$\phi_1 : \big(Y_1, \ldots, Y_n\big) \mapsto \big(\hat{S}_{1,1}, \ldots, \hat{S}_{1,n}\big)$$

$$\phi_2 : \big(Y_1, \ldots, Y_n\big) \mapsto \big(\hat{S}_{2,1}, \ldots, \hat{S}_{2,n}\big)$$

Which pairs (D_1, D_2) are achievable?

Some Remarks

1. Region depends on $|\rho|$. The sign of ρ is immaterial.
2. Distortion scales linearly with

$$(\sigma_1^2, \sigma_2^2).$$

3. Region is convex.
4. We shall thus assume

$$\sigma_1^2 = \sigma_2^2 \triangleq \sigma^2 \qquad \rho > 0.$$

We focus on the SYMMETRIC CASE where

$$P_1 = P_2 \triangleq P.$$

Achievability Results

- Uncoded Transmission
- Independent Gaussian Codebooks
- Time Sharing and Power Splitting $\Longrightarrow$ lower convex envelope
- Superposition of coded and uncoded transmission

Uncoded Transmission

$$X_{1,k} = \sqrt{\frac{P}{\sigma^2}} S_{1,k}, \quad k = 1, \ldots, n.$$

$$X_{2,k} = \sqrt{\frac{P}{\sigma^2}} S_{2,k}, \quad k = 1, \ldots, n.$$

$$Y_k = \sqrt{\frac{P}{\sigma^2}} S_{1,k} + \sqrt{\frac{P}{\sigma^2}} S_{2,k} + Z_k.$$

Use a linear estimator for $\hat{S}_{1,k}$ and $\hat{S}_{2,k}$:

$$D^*(\sigma^2, \rho, P, N) \leq \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}$$

Excellent for $\rho = 1$ but otherwise doesn't tend to zero as $P/N \to \infty$

A Lower Bound on D^*

For a memoryless bi-variate source and $d_1(s_1, \hat{s}_1), d_2(s_2, \hat{s}_2) \geq 0$ the pair (D_1, D_2) is achievable with powers P_1, P_2 only if

$$\min_{P_{\widehat{S}_1,\widehat{S}_2|S_1,S_2}} I(S_1, S_2; \widehat{S}_1, \widehat{S}_2)$$

$$\text{such that } \mathsf{E}\left[(S_1 - \widehat{S}_1)^2\right] \leq D_1,$$

$$\mathsf{E}\left[(S_2 - \widehat{S}_2)^2\right] \leq D_2,$$

does not exceed

$$\frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\rho_{\max}\sqrt{P_1 P_2}}{N}\right),$$

where $\rho_{\max} = \rho_{\max}(S_1, S_2)$ is the Hirschfeld-Gebelein-Rényi maximal correlation:

$$\rho_{\max} = \sup \mathsf{E}[g(S_1)h(S_2)] \tag{4}$$

where the supremum is over all functions f, g under which

$$\mathsf{E}[g(S_1)] = \mathsf{E}[h(S_2)] = 0 \qquad \mathsf{E}\left[g^2(S_1)\right] = \mathsf{E}\left[h^2(S_2)\right] = 1 \tag{5}$$

In the Symmetric Gaussian Case

For $P_1 = P_2 = P$ we obtain

$$D^*(\sigma^2, \rho, P, N) \geq \begin{cases} \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N} & \text{for } \frac{P}{N} \in \left(0, \frac{\rho}{1-\rho^2}\right] \\ \sigma^2 \sqrt{\frac{(1-\rho^2)N}{2P(1+\rho)+N}} & \text{for } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases}$$

On the Optimality of Uncoded Transmission

For

$$\frac{P}{N} < \frac{\rho}{1-\rho^2}$$

the bounds agree!

$$D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}, \qquad \text{if } \frac{P}{N} < \frac{\rho}{1-\rho^2}$$

For $P/N < \rho/(1-\rho^2)$ uncoded transmission is optimal!

Idea Behind the Proof

- Use the Hirschfeld-Gebelein-Rényi maximal correlation to upper bound

$$I(X_1 + X_2; Y).$$

- Use the Data Processing Inequality to use this upper bound to obtain an upper bound on

$$I(S_1, S_2; \hat{S}_1, \hat{S}_2).$$

- Use this upper bound and the distortion-rate function to obtain a necessary condition on

$$(D_1, D_2).$$

Deriving an upper bound on $I(\mathbf{X}_1 + \mathbf{X}_2; \mathbf{Y})$

- By simple algebra

$$\begin{aligned}\mathsf{Var}[X_{1,k} + X_{2,k}] &= \mathsf{Var}[X_{1,k}] + \mathsf{Var}[X_{2,k}] + 2\rho(X_{1,k}, X_{2,k})\sqrt{\mathsf{Var}[X_{1,k}]} \cdot \sqrt{\mathsf{Var}[X_{2,k}]} \\ &\leq \mathsf{Var}[X_{1,k}] + \mathsf{Var}[X_{2,k}] + 2\tilde{\rho}_{\max}\sqrt{\mathsf{Var}[X_{1,k}]} \cdot \sqrt{\mathsf{Var}[X_{2,k}]}\end{aligned}$$

  where $\tilde{\rho}_{\max}$ is the maximum correlation between any functional of $(S_{1,1}, \ldots, S_{1,n})$ and any functional of $(S_{2,1}, \ldots, S_{2,n})$.

- Our bi-variate source is IID so $\tilde{\rho}_{\max}$ is the maximum correlation between any two functionals of $S_{1,1}$ and $S_{2,1}$ (Witsenhausen'75)

$$\mathsf{Var}[X_{1,k} + X_{2,k}] \leq \mathsf{Var}[X_{1,k}] + \mathsf{Var}[X_{2,k}] + 2\rho_{\max}\sqrt{\mathsf{Var}[X_{1,k}]} \cdot \sqrt{\mathsf{Var}[X_{2,k}]}$$

- Summing the variances we obtain using the Cauchy-Schwarz inequality

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{Var}[X_{1,k} + X_{2,k}] \leq P_1 + P_2 + 2\rho_{\max}\sqrt{P_1 P_2}.$$

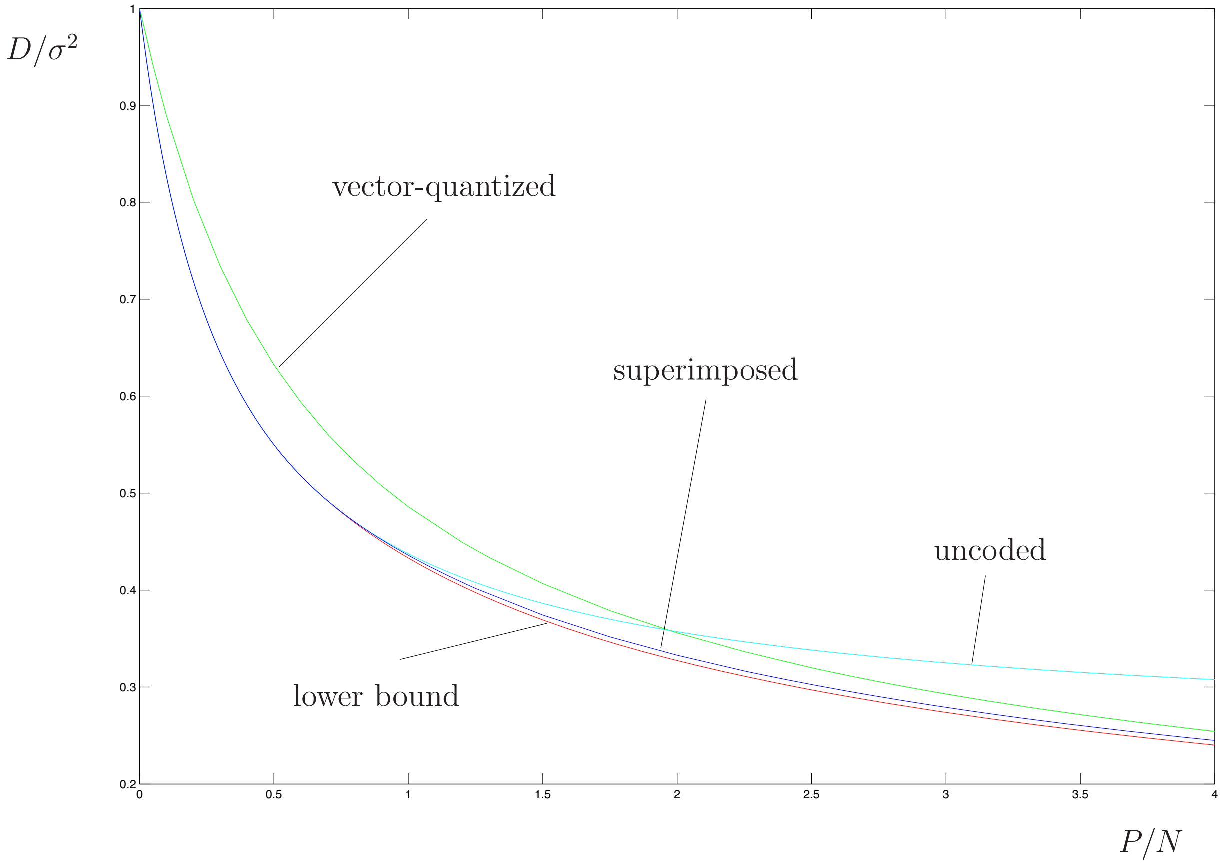
- But IID Gaussians maximize differential entropy subject to sum of variances, so

$$\frac{1}{n}h(\mathbf{Y}) \leq \frac{1}{2}\log\left(2\pi e\Big(P_1 + P_2 + 2\rho_{\max}\sqrt{P_1 P_2}\Big)\right)$$

- and hence

$$\frac{1}{n}I(\mathbf{X}_1 + \mathbf{X}_2; \mathbf{Y}) \leq \frac{1}{2}\left(1 + \frac{P_1 + P_2 + 2\rho_{\max}\sqrt{P_1 P_2}}{N}\right)$$

Some Bounds for $\rho = 0.4$



A. Lapidoth and S. Tinguely, "Sending a Bi-Variate Gaussian Source over a Gaussian MAC," to be presented at ISIT'06.

Some High SNR Asymptotics

$$D^* \approx \sigma^2 \cdot \sqrt{\frac{N}{P}} \cdot \sqrt{\frac{1-\rho}{2}}$$

in the sense that the ratio of the two sides tends to one.