Parity Forwarding for the Relay Network

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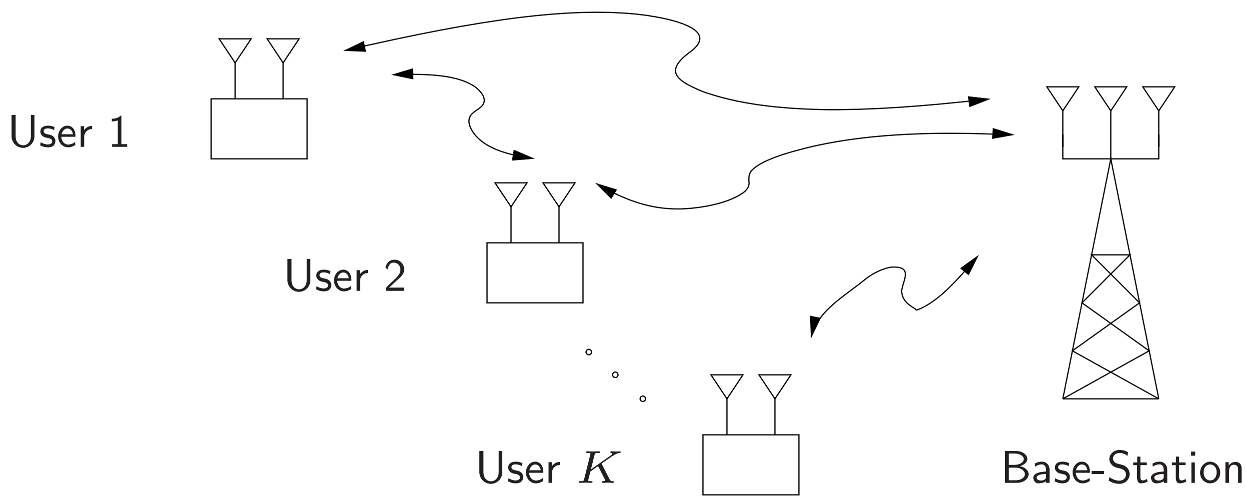
Relay Network



- Information Theory: Cover and El Gamal ('79) – Binning strategy
- Communication Strategies: Decode-and-forward, Amplify-and-forward.

Outline of This Talk

- Is binning practical for the relay channel?
  - Part I: LDPC code design to approach the DF capacity.
- How to generalize binning to multi-relay networks?
  - Part II: Binning schemes for multi-relay networks.

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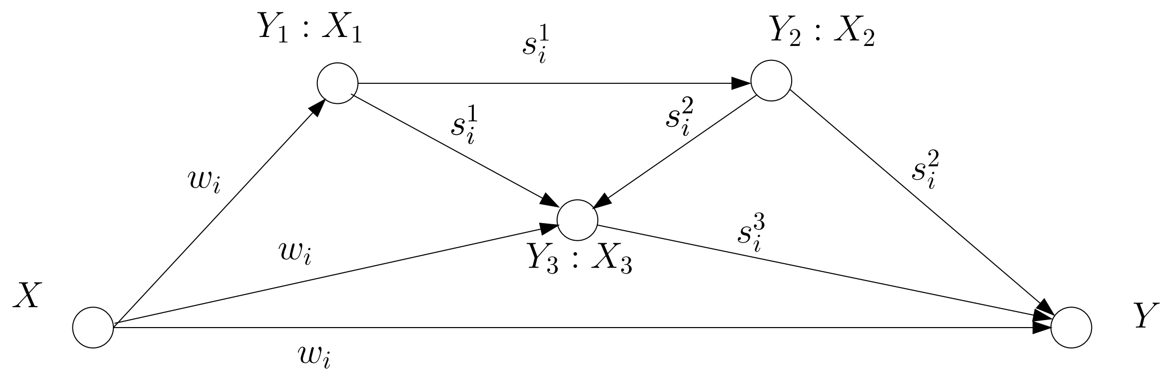
Binning via "Parity Forwarding"

Information Flow in a Relay Network



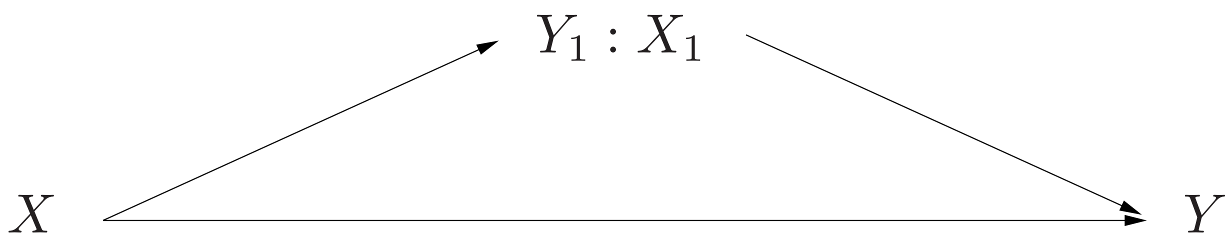
- Relay nodes help communication via decoding-and-retransmit-parities.

- Design challenges:
  - Routing of information in a network.
  - Efficient codes to facilitate decoding at the relays/destination.

Part I: Is binning practical for the relay channel?

Binning in Decode-and-Forward

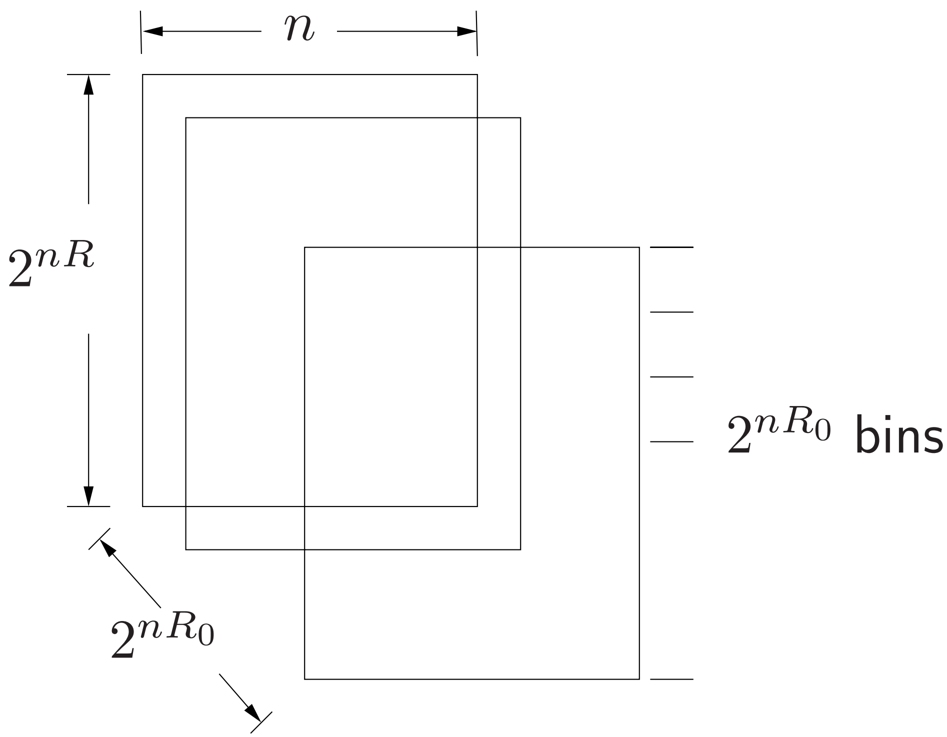
- Consider Cover and El Gamal's strategy for degraded relay channel:



- Two elements: Block-Markov coding and Binning
  - The relay provides a bin index of the transmitter codeword.

$$C = \sup_{p(x,x_1)} \min\{I(X, X_1; Y), I(X; Y_1|X_1)\}$$

Code Construction



Doubly indexed codebook $X^n(w|s)$

$w \in \{1, 2, \cdots, 2^{nR}\}$

$s \in \{1, 2, \cdots, 2^{nR_0}\}$

2^{nR} is partitioned into 2^{nR_0} bins

Second Codebook $X_1^n(s)$

What is binning?

- Binning is ubiquitous in multiuser information theory
  - Writing on dirty paper (Gel'fand-Pinsker)
  - Source coding with encoder side information (Wyner-Ziv)
  - Relay communication (Cover-El-Gamal)
- Binning is a way of conveying "partial" information.

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Bin index is equivalent to parity-checks

Bin Index as Parity-Checks

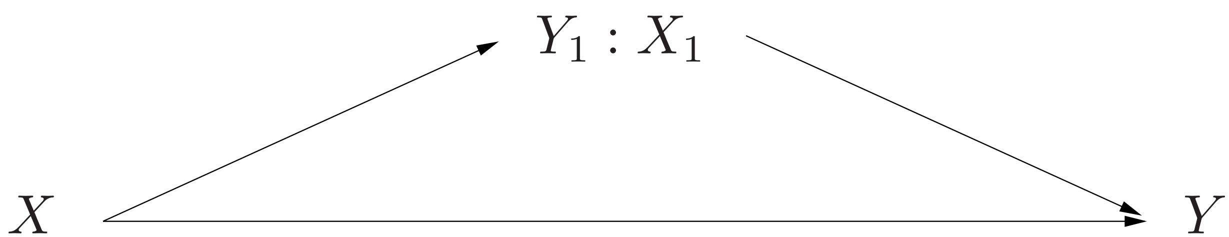
How do we partition a codebook of size 2^{nR} into 2^{nR_0} bins?

... by forming nR_0 parity check bits,
and using the parity check bits as bin indices.

Same idea as DISCUS for Slepian-Wolf coding (Pradhan-Ramchandran)
or structured binning (Zamir-Shamai-Erez)

Decode and Forward

- X_1 decodes X and re-encodes parities (or a bin index) of X.



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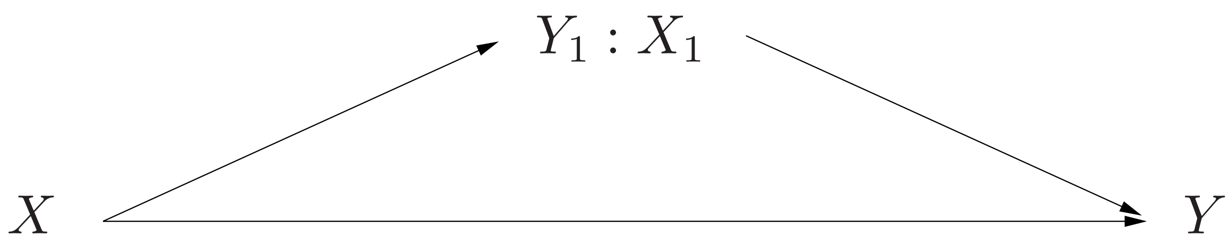
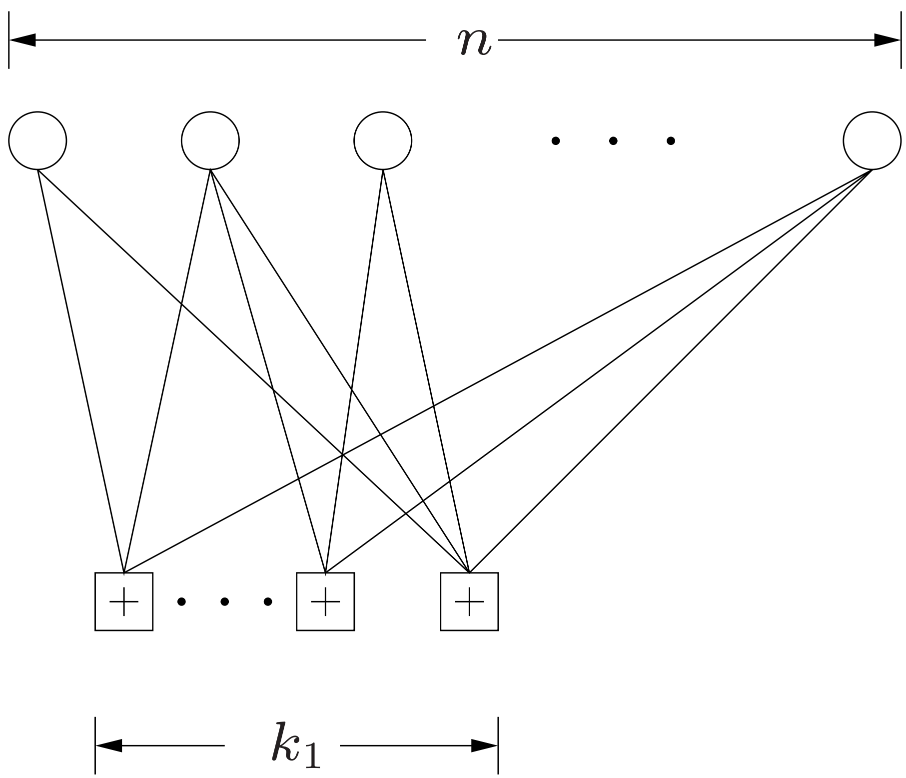


- A good code for the relay channel must be capacity-approaching
  - for the $X - Y_1$ link at R;
  - for the $X - Y$ link at $R - R_0$ with extra parities!

Bi-Layer LDPC Code for the Relay Channel



(n, k_1) must be capacity approaching for $X - Y_1$

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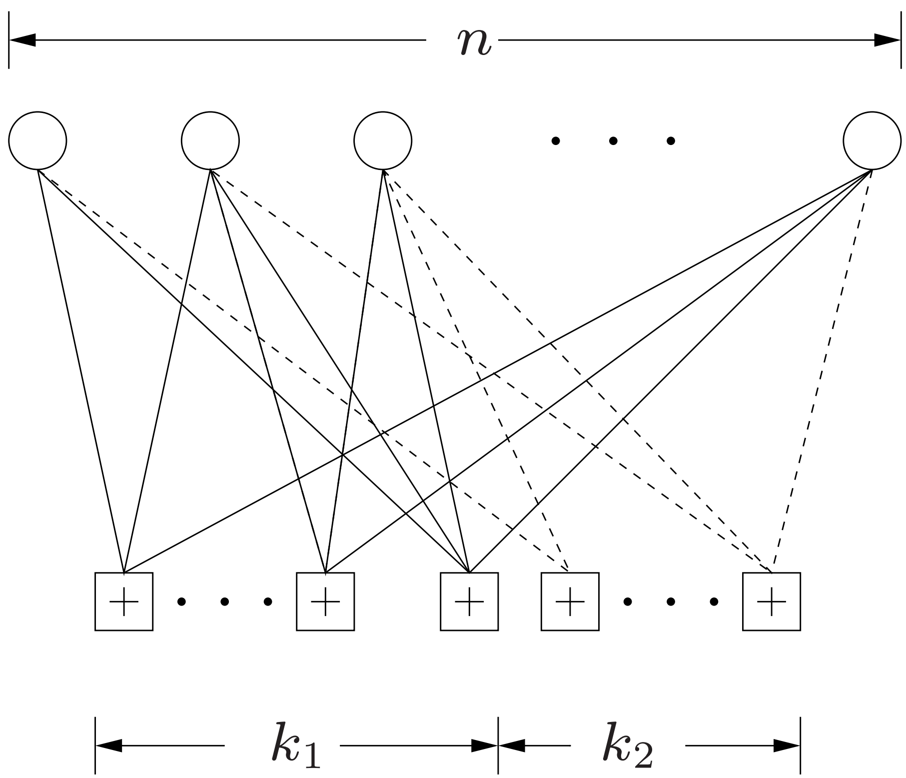


(n, k_1) must be capacity approaching for $X - Y_1$

$(n, k_1 + k_2)$ is capacity approaching for $X - Y$

Bi-Layer LDPC Code

Code Design Problem for the Relay Channel

Design a single LDPC code so that:

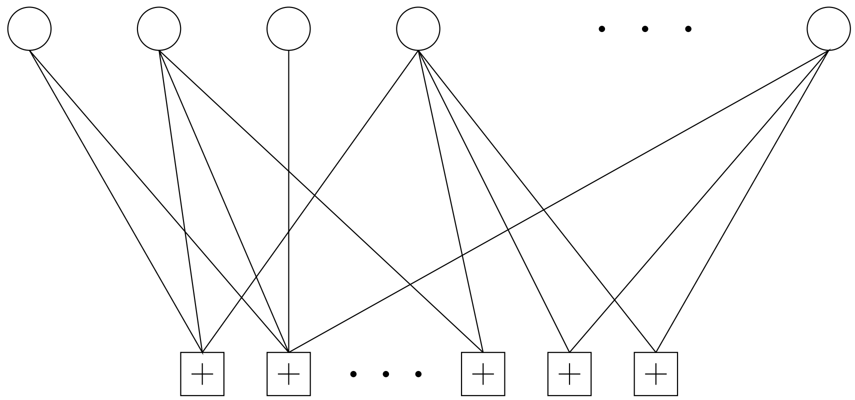
- The entire graph is capacity-achieving at $R - R_0$ with SNR_{low}.
- The sub-graph is capacity-achieving at R with SNR_{high}.

Objective of this part of the talk:

Design degree sequence for an LDPC code to achieve the above.

Universal coding problem!

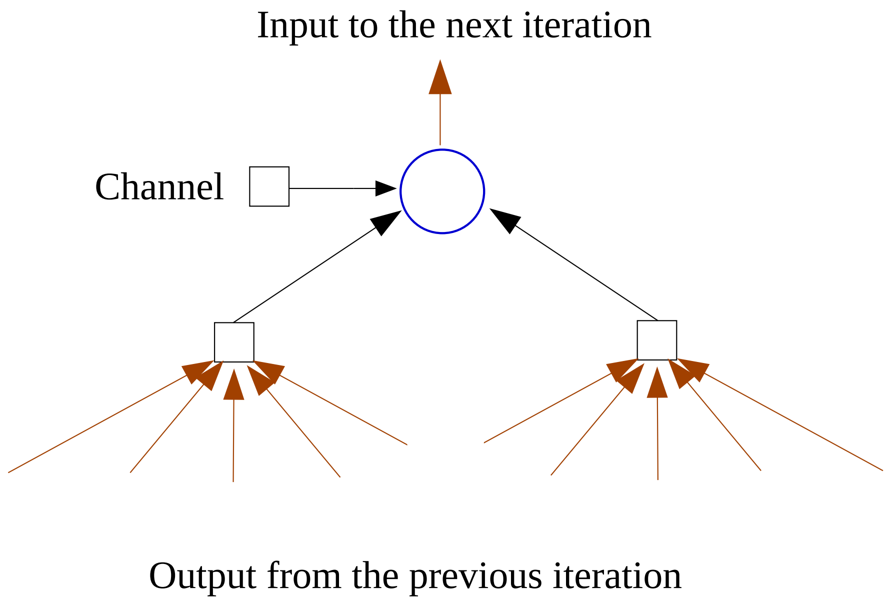
Irregular Low-Density Parity-Check Codes



- An ensemble of irregular LDPC codes is defined by its variable-degree distribution $\{\lambda_2, \lambda_3, \ldots\}$ and its check-degree distribution $\{\rho_2, \rho_3, \ldots\}$.

- Degree distribution is related to rate by: $\boxed{R = 1 - \frac{\sum_i \frac{\rho_i}{i}}{\sum_i \frac{\lambda_i}{i}}}$

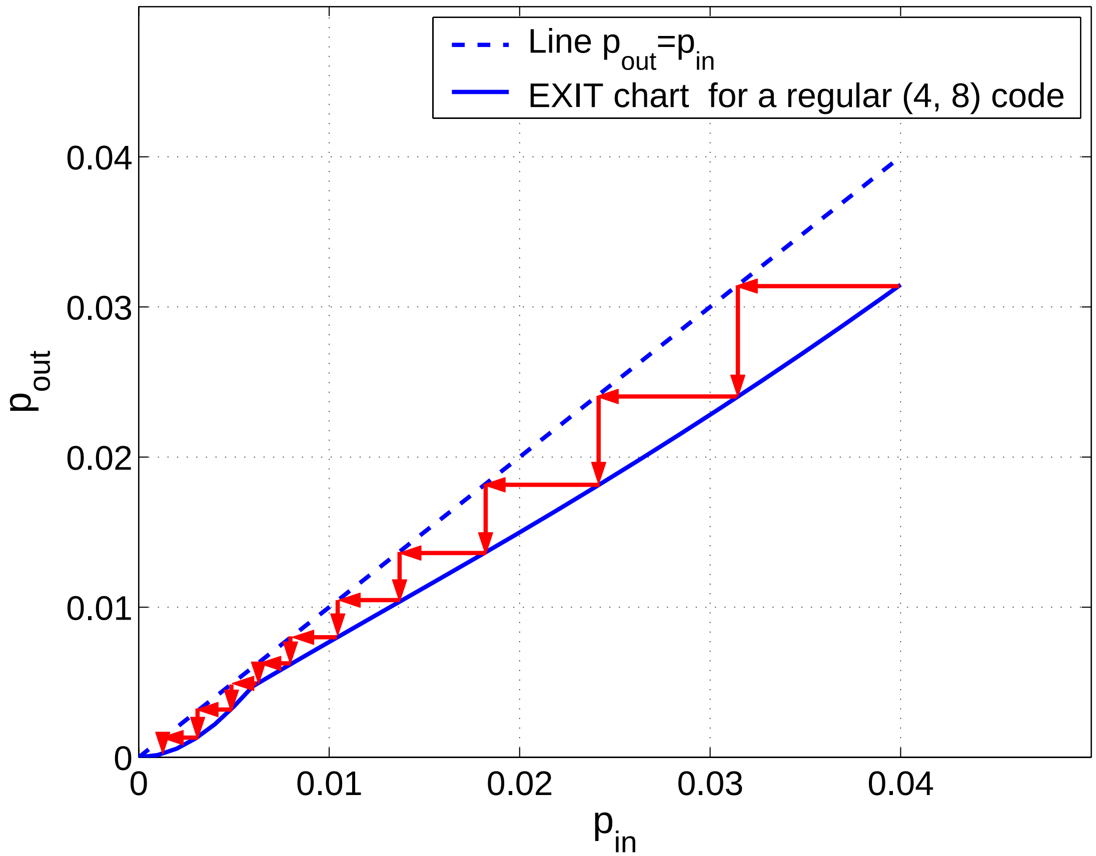
Iterative Decoding Algorithm



- A message is a belief about the incident variable node
- Decoder passes messages between check and variable nodes iteratively.

- Analysis Tool: Density Evolution (Urbanke-Richardson)
- This talk: Extrinsic Information Transfer (EXIT) charts (ten Brink)

Tracking Extrinsic Probability of Error



EXIT Chart for BSC with $\epsilon = 0.04$

- Mutual Inform. EXIT Chart (ten Brink '01)
- Prob. of Error EXIT Chart (Ardakani, Kschischang '04)

Shaping the EXIT Chart

- For an irregular LDPC code, P_{out} at the output of variable nodes is computed using Bayes's rule.
- Assume a fixed check degree distribution, the resulting P_{out} is equivalent to a linear combination of corresponding P_{out} of regular codes
- Therefore, the EXIT chart of an irregular code is a linear combination of elementary EXIT charts of regular codes, making P_e-EXIT chart a powerful design tool.

$$\boxed{f(p) = \sum_i \lambda_i f_i(p)}$$

Linear Programming Approach to LDPC Code Design

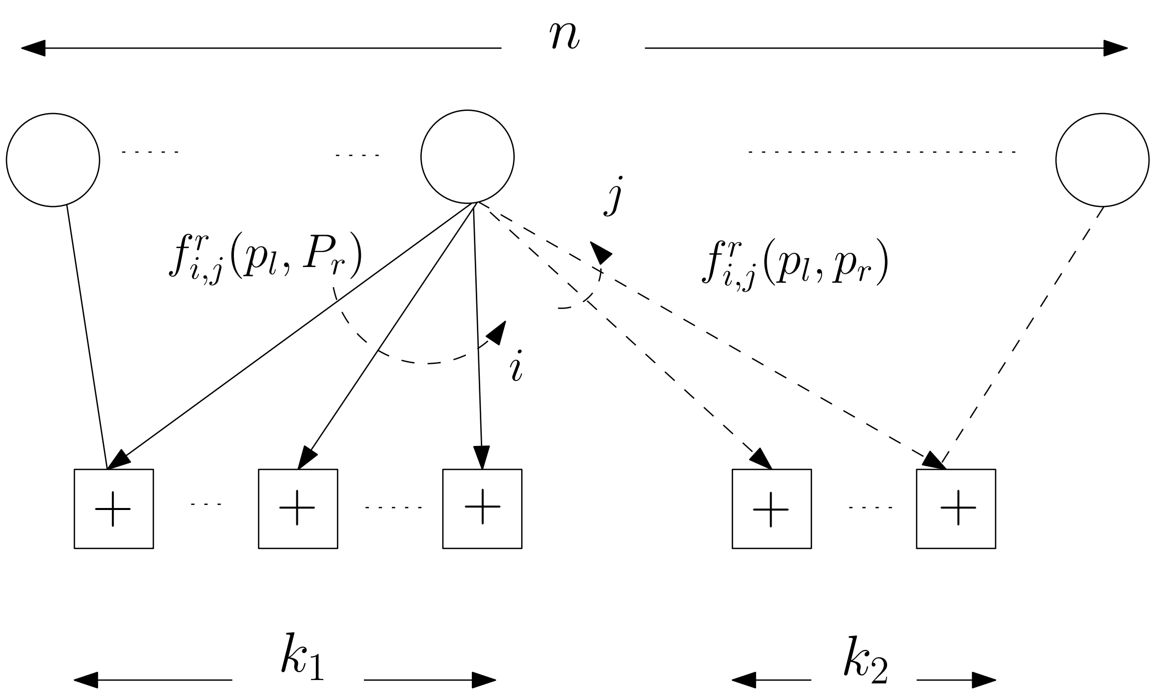
- Consider a code design problem for a standard BSC or AWGN channel:
  - Fix check degree sequence ρ_i.

$$\begin{aligned} \text{maximize} \quad & 1 - \frac{\sum \rho_i / i}{\sum \lambda_i / i} \\ \text{subject to} \quad & \sum \lambda_i f_i(p) < p \end{aligned}$$

  - Choose variable degree sequence λ_i to maximize rate, subject to decodability constraints, by solving a *linear programming* problem.

- This talk: Generalizing this approach to design *bi-layer* codes.

Bi-Layer Density Evolution



Keep track of the probability of error in left and right graphs (p_l, p_r).
Define left and right elementary EXIT charts $f^l_{i,j}(p_l, p_r)$ and $f^r_{i,j}(p_l, p_r)$.

Designing Bi-Layer LDPC Codes for the Relay Channel

$$
\begin{aligned}
\text{maximize} \quad & 1 - \frac{\sum_i \rho_i / i}{\sum_i \nu_i / i} \\
\text{subject to} \quad & \nu_i = \frac{1}{\eta} \sum_j \frac{i}{i+j} \lambda_{i,j} \\
& \sum_i \nu_i f_i^s(p) < p \\
& \sum_{i,j} \lambda_{i,j} \frac{f_{i,j}^l(p_l, p_r) i + f_{i,j}^r(p_l, p_r) j}{i+j} < \eta p_l + (1-\eta) p_r
\end{aligned}
$$

The design variables are left and right degree sequences $\lambda_{i,j}$.

Performance

Optimal $\lambda_{i,j}$ (left degree i and right degree j) for a relay channel with $R_{source-relay} = 0.7520$ and $R_{source-destination} = 0.6280$.

(i,j)	$j=0$	$j=1$	$j=2$	$j=3$
$i=2$	0.1153	0.0623	0	0
$i=3$	0.1220	0.0921	0	0
$i=5$	0	0.1897	0	0
$i=8$	0	0	0.0591	0
$i=9$	0	0	0.0166	0
$i=20$	0	0	0.3296	0.0132

Gap to capacity: **0.19dB** for source-relay, **0.34dB** for source-destination.

Comparing with Other Work

- For fading relay channel, previously reported gap is about 1-1.5dB
  - Zhao-Valenti '03
  - Zhang-Bahceci-Duman '04
  - Khojastepour-Ahmed-Aazhang '04

- After the workshop, we also become aware of very recent independent work based on density evolution:
  - Chakrabarti, de Baynast, Sabharwal, Aazhang '06

How Hard is Binning?

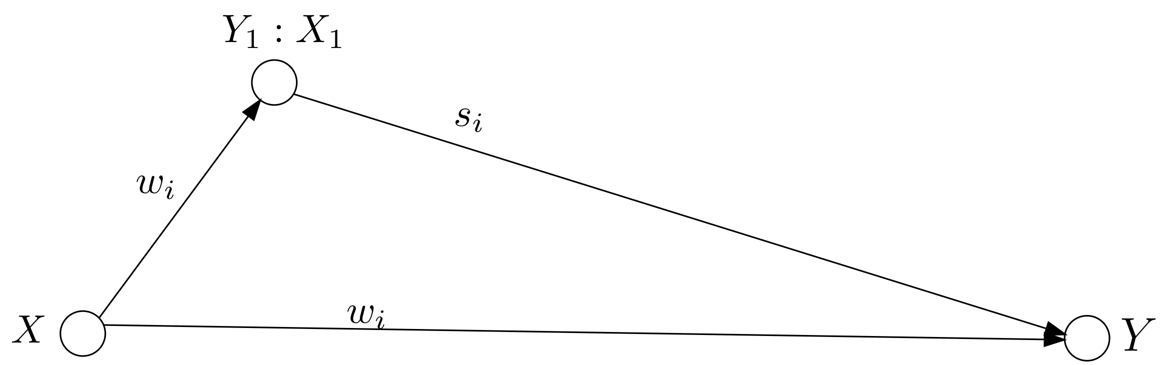
- Implementing binning:
  - Binning for quantization is hard. (e.g. Gel'fand-Pinsker, Wyner-Ziv)
  - Binning for error-correcting is practical! (e.g. DF in relay channel)
- Main message:

$$\boxed{\text{Binning for Relay Channel} = \text{Parity Forwarding}}$$

- The coding problem $\Rightarrow$ Designing a *universal* code.

Part II: Parity-Forwarding for Multi-Relay Networks

Parity Forwarding for One-Relay Network

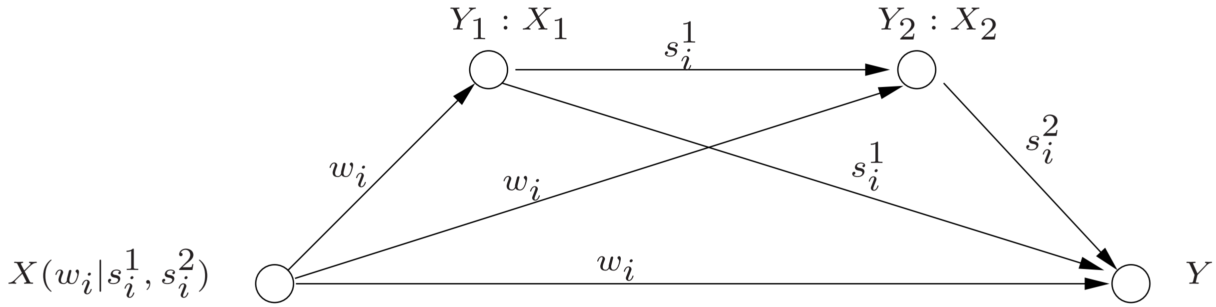


Key equations for Cover-El-Gamal strategy:

$$\begin{aligned} R &< I(X;Y_1|X_1) && \text{decodability at the relay} \\ R_0 &< I(X_1;Y) && \text{parity-forwarding from relay to destination} \\ R - R_0 &< I(X;Y|X_1) && \text{final decoding at the destination} \end{aligned}$$

"Degraded" means that relay is able to decode the source message.

Two-Relay Network

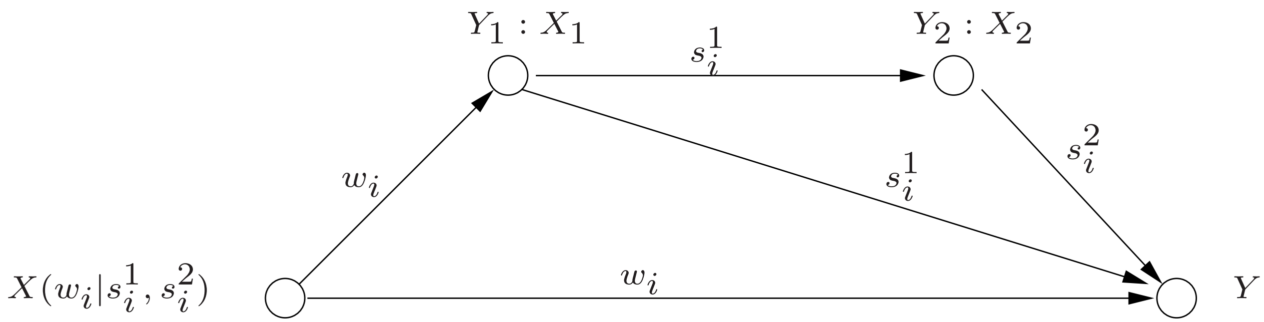


- What does degradedness mean for multi-relay networks?
  - Both relays are capable of decoding the source message. – Proof via regular encoding. (Xie-Kumar'05, Kramer-Gastpar-Gupta'05)

$$C = \max_{p(x,x_1,x_2)} \min\{I(X;Y_1|X_1,X_2), I(X,X_1;Y_2|X_2), I(X,X_1,X_2,;Y)\}$$

  - We call the above *serially degraded* relay channel.

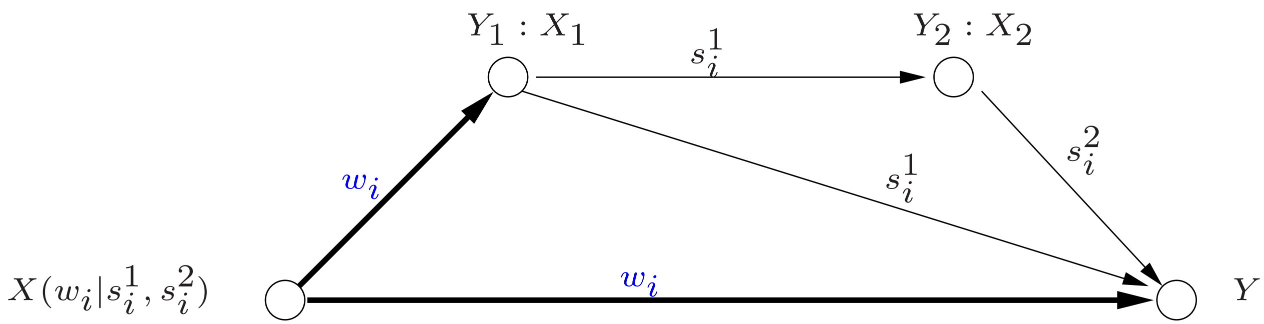
Another Case: Doubly Degraded Two-Relay Network



- Suppose that the link from source to the second relay is weak:
  - We do not require the second relay to decode the source message.
  - But, we use the second relay to help the first relay transmit the help-message to the destination.

- We call this a *doubly degraded* relay network.

Doubly Degraded Two-Relay Network



- Four-step block-Markov coding:
  - Source transmits w_i to both Y_1 and Y.
  - First relay decodes w_i and transmits s_i^1 (parities of w_{i-1}) to Y_2, Y.
  - Second relay decodes s_i^1 and transmits s_i^2 (parities of s_{i-1}^1) to Y.
  - Destination decodes s_i^2 first, then s_{i-1}^1, finally w_{i-2}.

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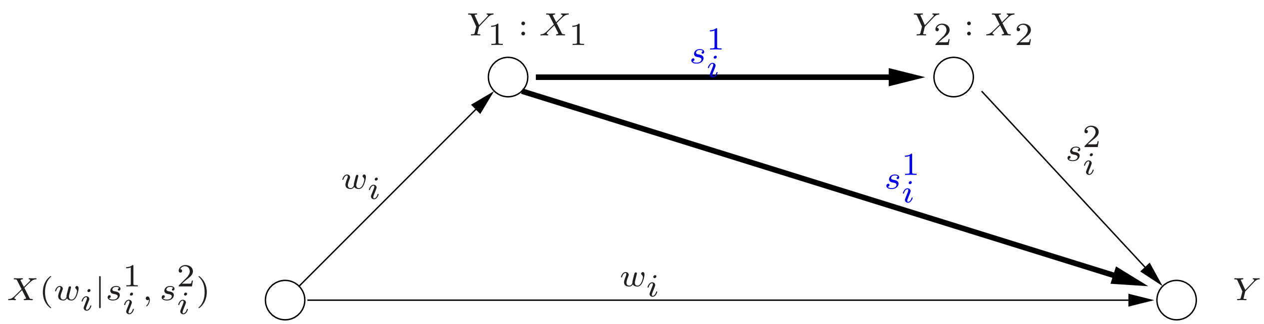

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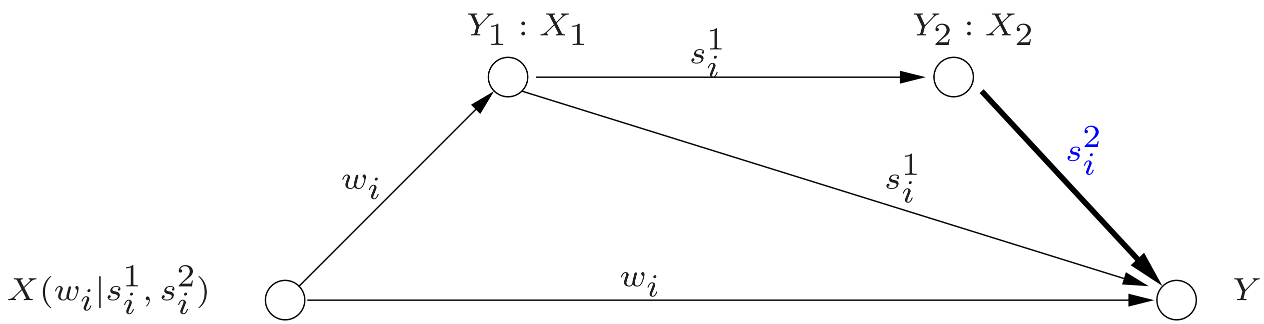

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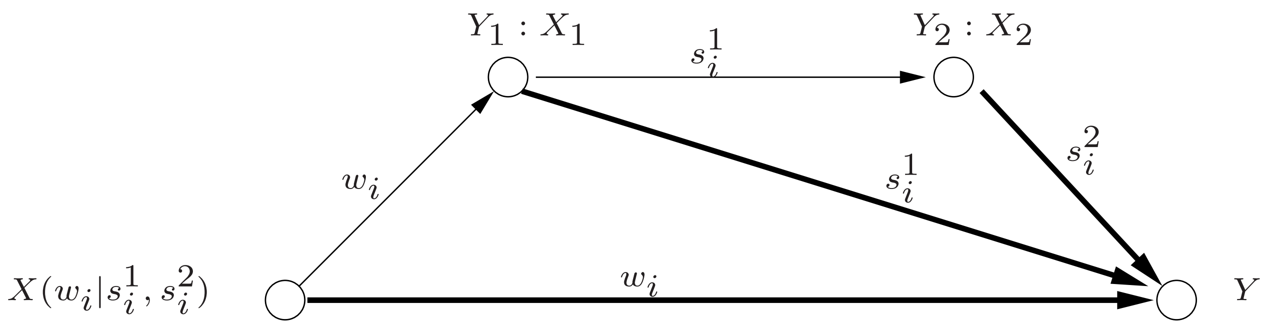

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Capacity for Doubly Degraded Two-Relay Network

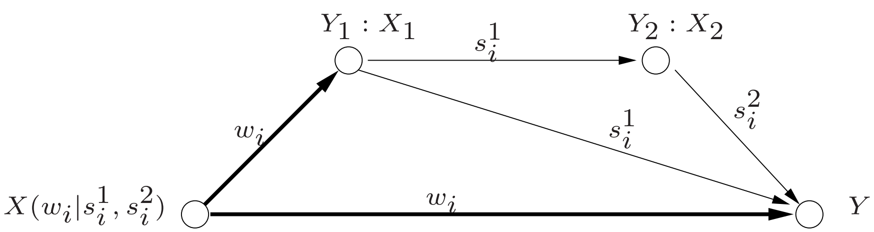
Definition 1. *A doubly degraded two-relay network is defined by* $p(y, y_1, y_2|x, x_1, x_2)$*, where* $X - (X_1, X_2, Y_1) - (Y_2, Y)$*,* $X_1 - (X_2, Y_2) - Y$ *and* $X - (X_1, X_2, Y) - Y_2$ *form Markov chains.*

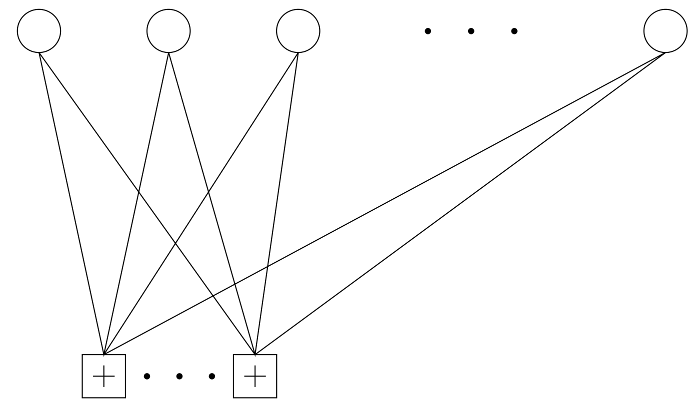
Theorem 1. *The following rate maximized over* $p(x, x_1, x_2)$ *is achievable*

$$\begin{aligned} R &< I(X; Y_1|X_1, X_2). \\ R &< I(X; Y|X_1, X_2) + I(X_1; Y_2|X_2) \\ R &< I(X; Y|X_1, X_2) + I(X_1; Y|X_2) + I(X_2; Y) \\ &= I(X, X_1, X_2; Y). \end{aligned}$$

It is also the capacity if the two-relay network is doubly degraded.

Coding for Doubly Degraded Relay Network





Cascade of bi-layer codes!

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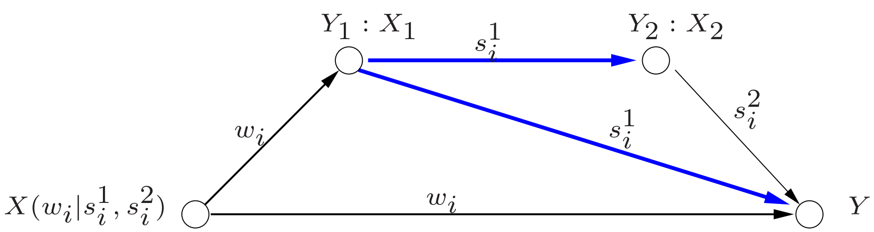

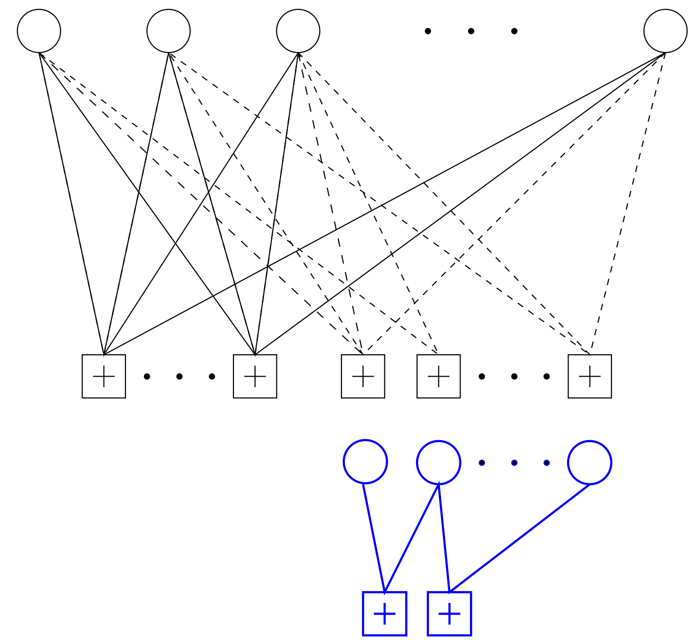

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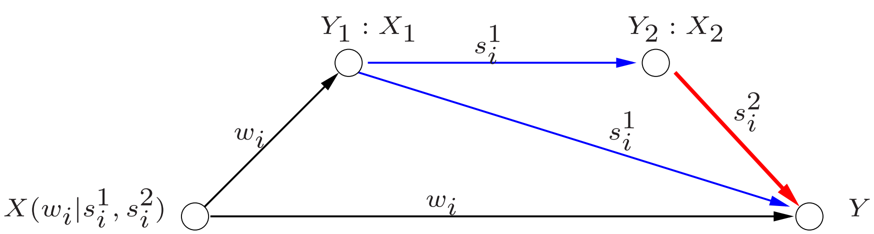
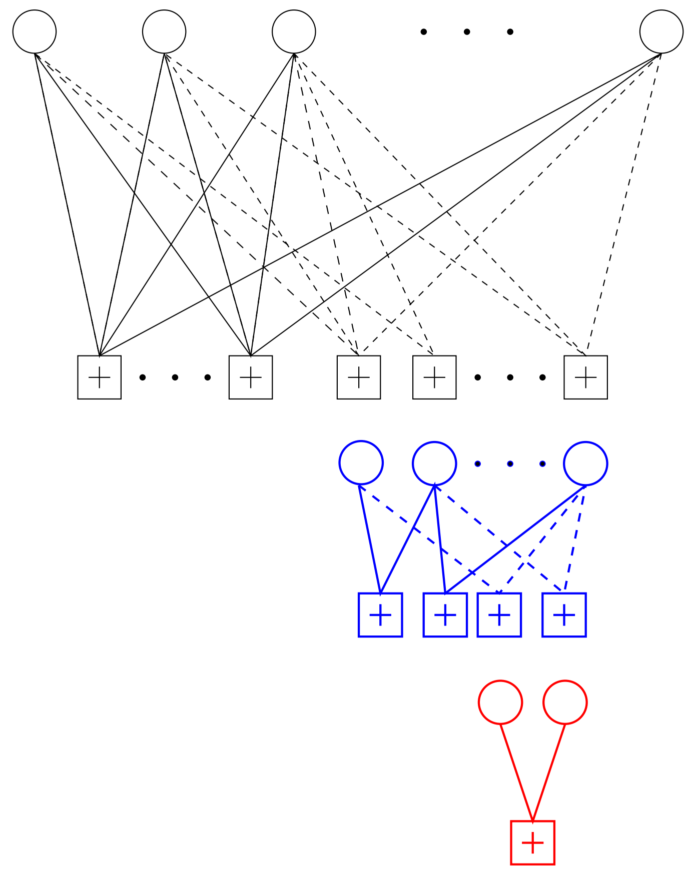
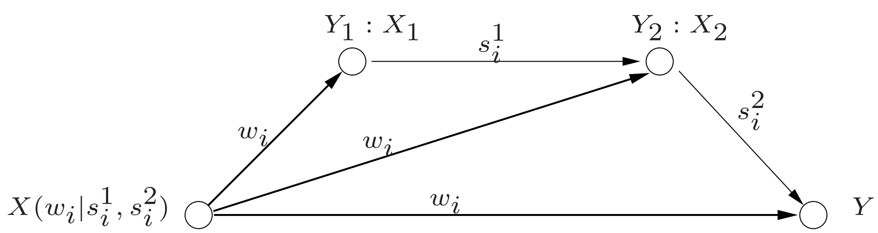
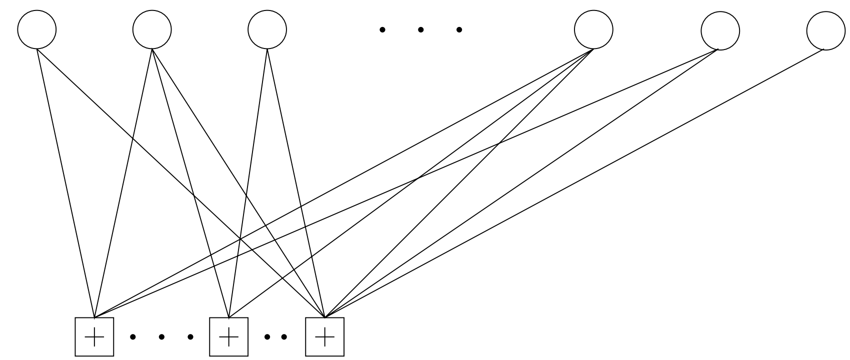




Cascade of bi-layer codes!

Another Case: Tri-Layer LDPC Codes





Embedded tri-layer code!

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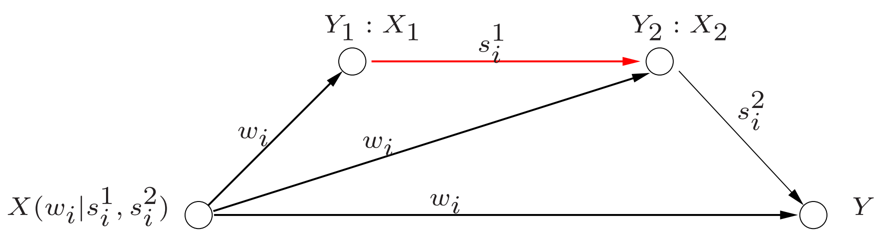

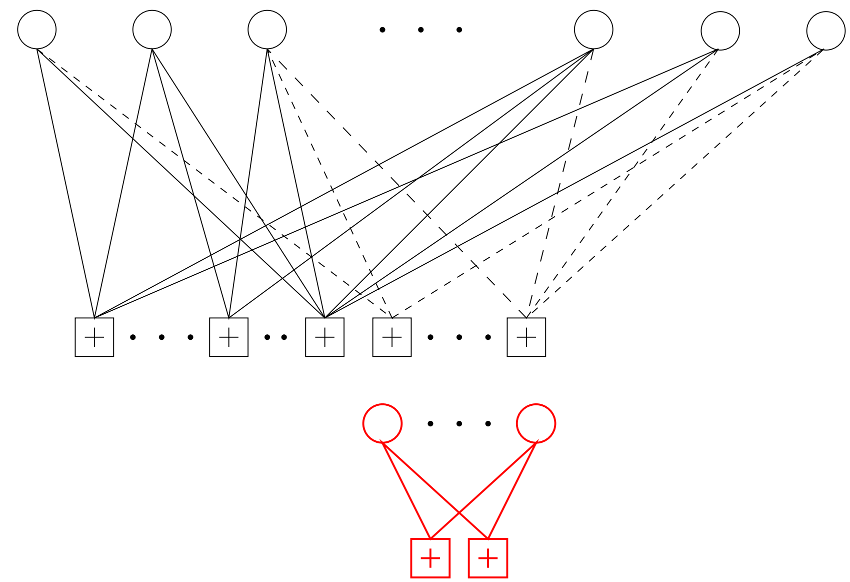

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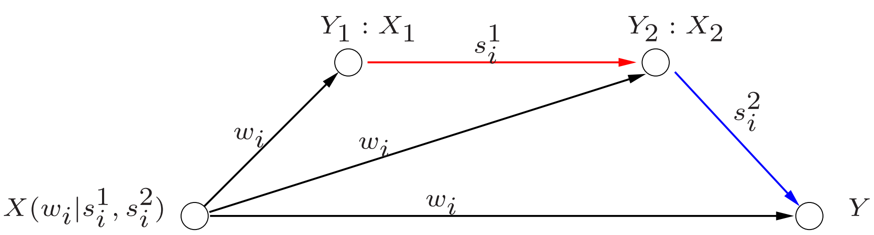

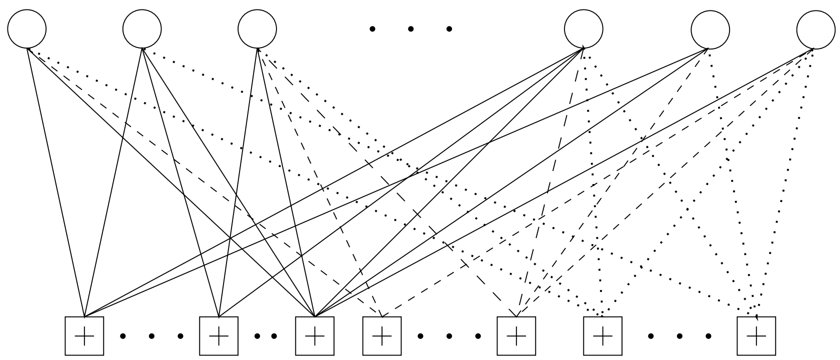

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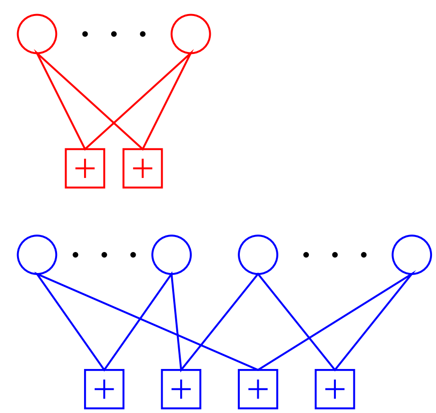
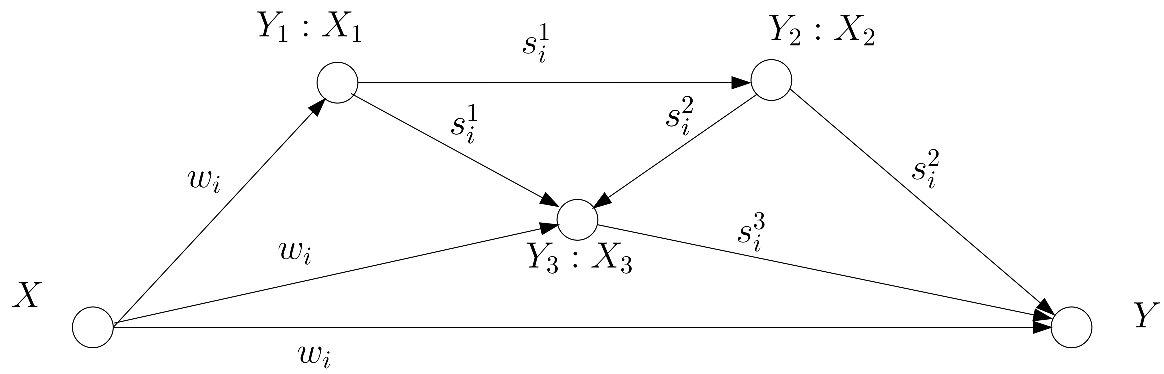


General Relay Networks



- The order at which different nodes help each other can be visualized:
  - Node X_1 helps Node Y_3 to decode w_i by sending s_i^1.
  - Node X_2 helps Node Y_3 to decode s_i^1 by sending s_i^2.
  - Node X_3 helps the destination in decoding both s_i^2 and w_i.

- This is like a routing protocol! Coding problem: *universal* codes!

Concluding Remarks

- This talk gives new interpretation and insights on relay strategies:
  - Existing relay protocols can be interpreted as parity-forwarding.
  - Parity-forwarding can be efficiently implemented using LDPC codes.
  - Multi-relay networks can be degraded in more than one way; parity-forwarding is capacity-achieving in degraded networks.
- Connection with Fountain codes and Network coding:
  - Parity-generation achieves universal coding in an erasure network.
  - Parity-formation achieves maximum single-source multicast throughput in network coding.
  - Parity-forwarding achieves decode-and-forward rate in relay networks!