On Achievable Information Rates in Single-Source Non-Uniform Demand Networks

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Simple Universal Characterizations of Achievable Information Rates

over lossless networks

Min-Cut Max-Flow Theorem

Consider a network represented as a graph with unit-capacity edges, h unit-rate information sources S_1, …, S_h and one receiver. Assume the min-cut to the receiver is h.

[Ford-Fulkerson] ~1950

There exist h edge-disjoint paths from the sources to the receiver.

Main Theorem in Network Coding

Consider a network represented as a directed graph with unit-capacity edges, h unit-rate information sources $S_1, \ldots, S_h$ located on the same vertex of the graph and N receivers $R_1, \ldots, R_N$. Assume the min-cut to each receiver is h.

[Alshwede,Cai,Li,Yeung] ~2000

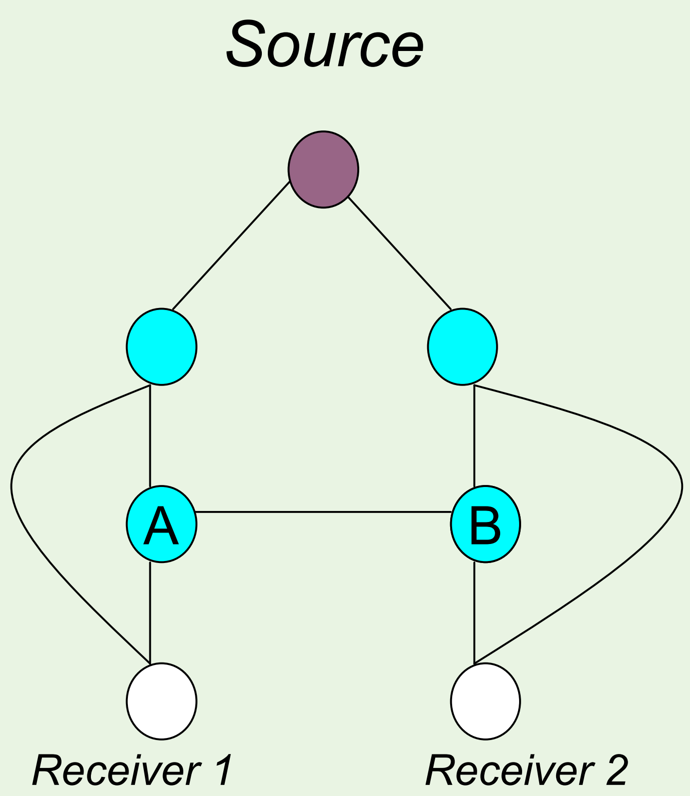
We can simultaneously transmit rate h to all receivers if intermediate nodes in G can linearly re-encode information.

"The network is solvable"

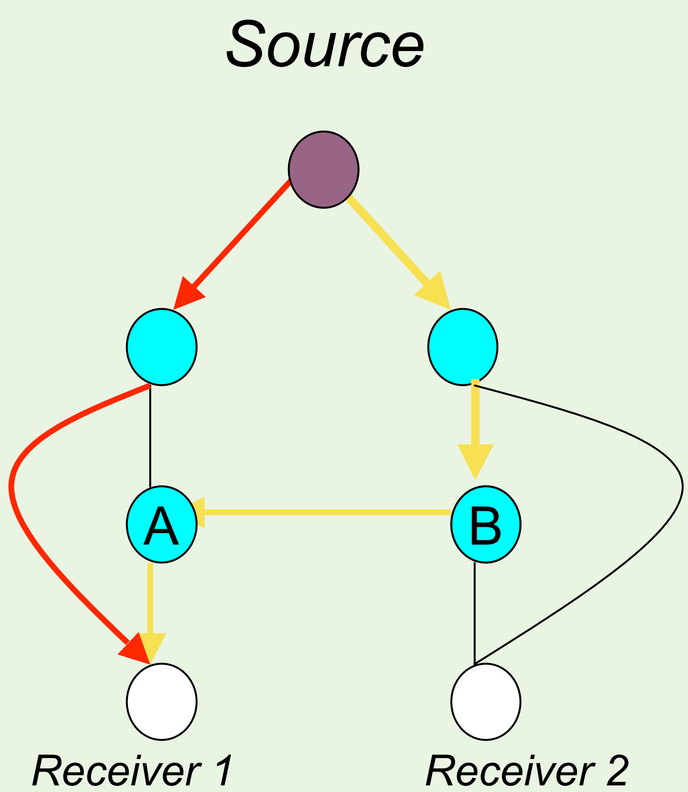
Undirected Graphs

Consider a network represented as an undirected graph with unit-capacity edges, h unit-rate information sources $S_1, \ldots, S_h$ located on the same vertex of the graph and N receivers $R_1, \ldots, R_N$. Assume the min-cut to each receiver is h.

Undirected Graphs



Undirected Graphs



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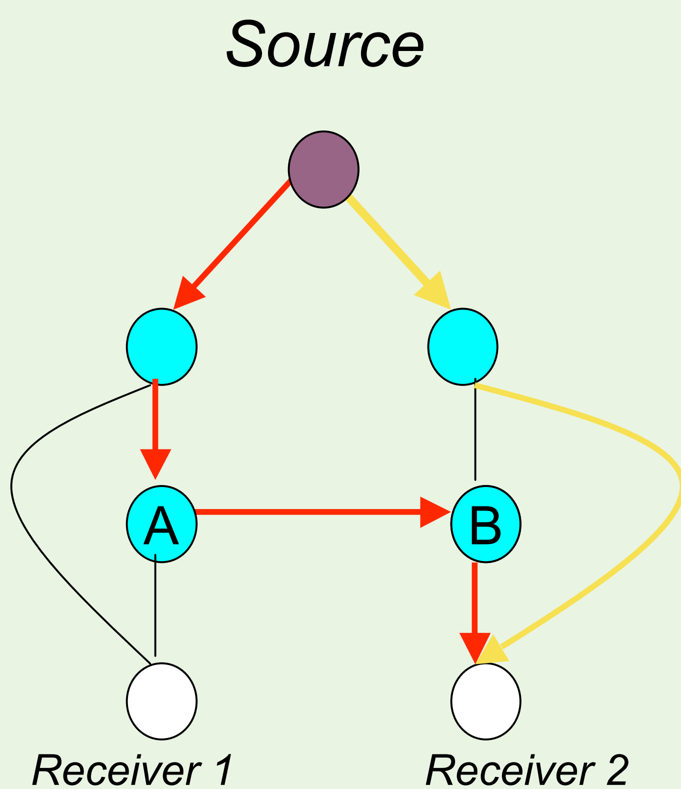

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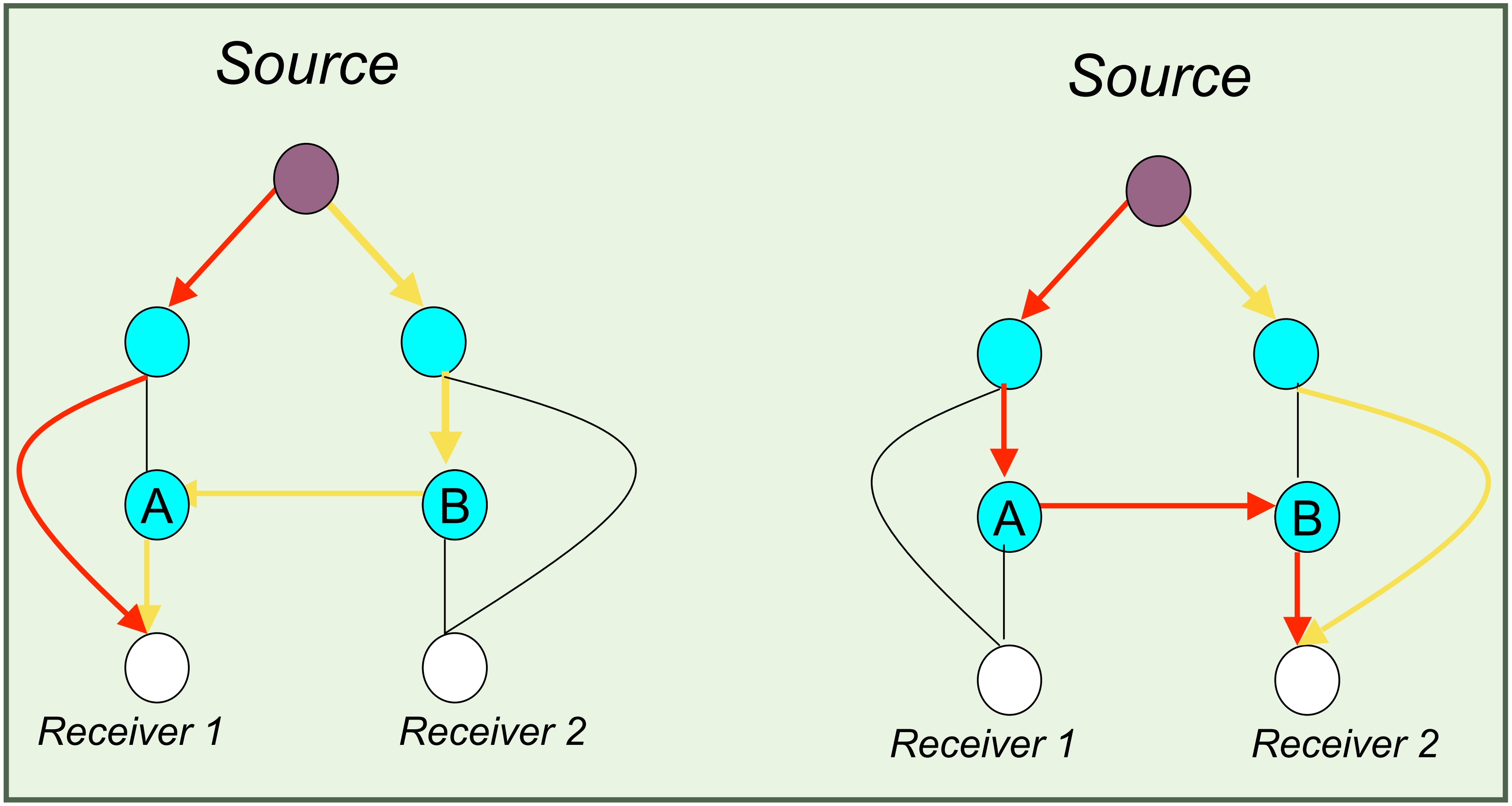

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Not in general solvable

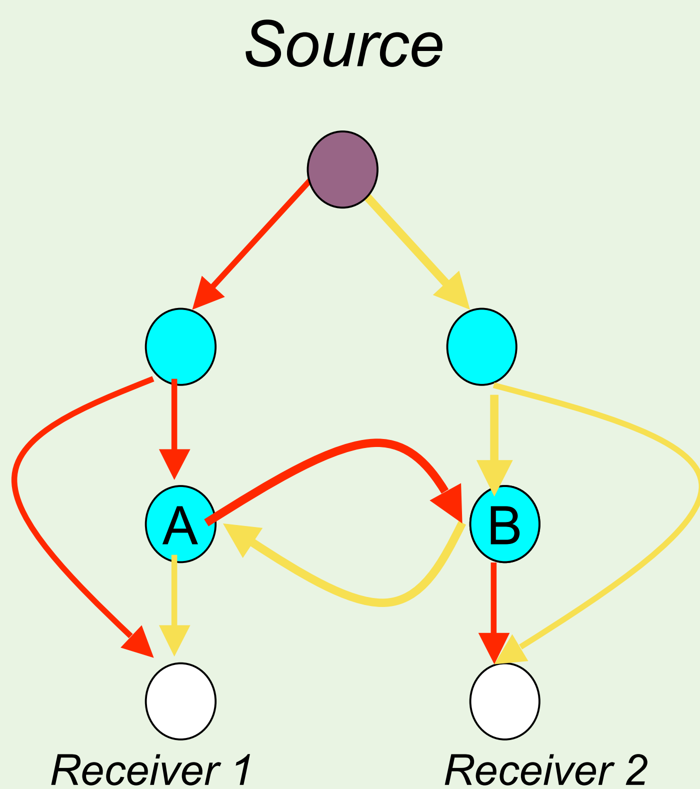
Undirected Graphs

Consider a network represented as an undirected graph with unit-capacity edges, h unit-rate information sources $S_1, \ldots, S_h$ located on the same vertex of the graph and N receivers $R_1,\ldots,R_N$. Assume the min-cut to each receiver is h.

[Li and Li] ~2003

We can simultaneously transmit rate h/2 to all receivers, even when only using routing.

Undirected Graphs

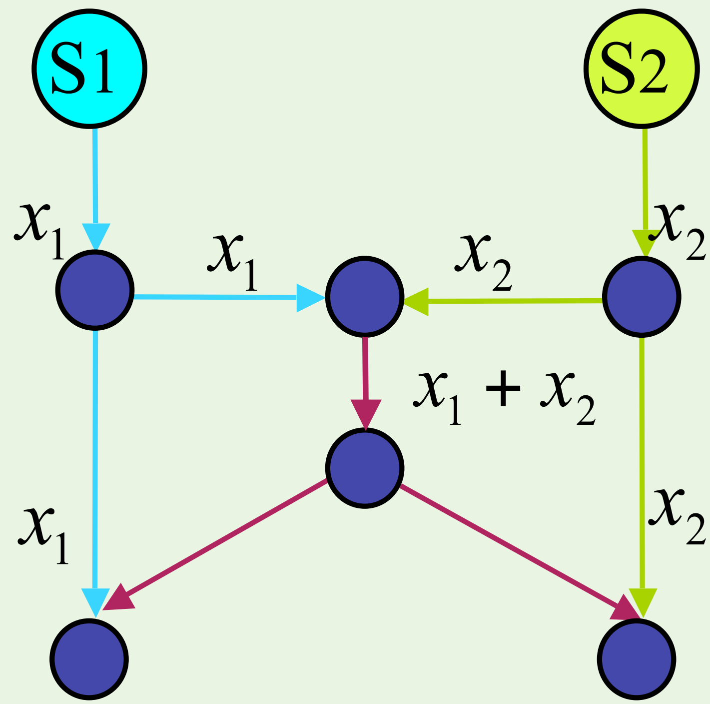


Different Min-Cut

Consider a network represented as a graph with unit-capacity edges, h unit-rate information sources $S_1, \ldots, S_h$ and N receivers $R_1,\ldots,R_N$. The min-cut to each receiver might be different.

Not in general solvable

Different Min-Cut



Every node of the network is a receiver

Different min-cuts
Specific Demand Networks

Each receiver demands to receive a specific subset of the sources.

Different min-cuts
Specific Demand Networks

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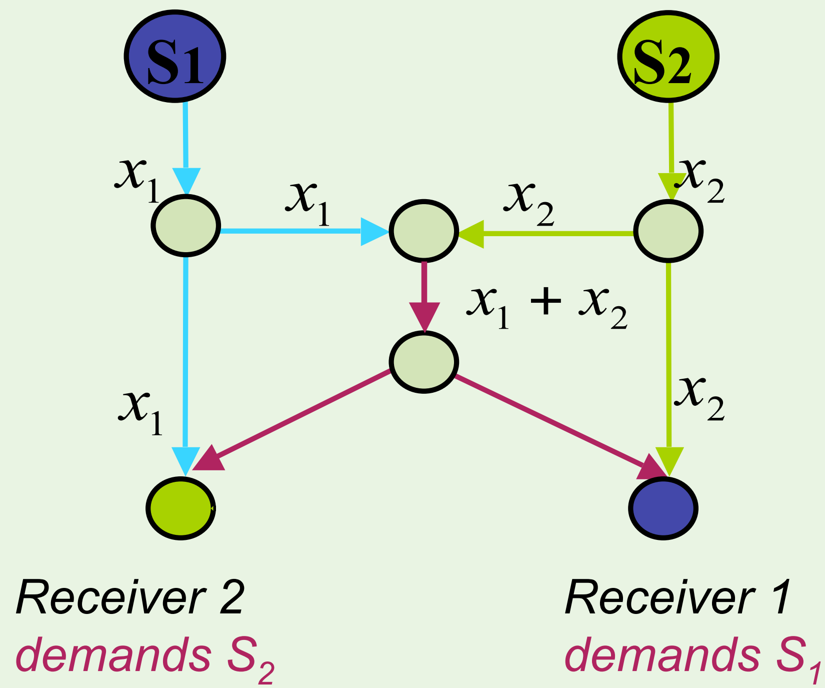

Different min-cuts
Specific Demand Networks

Each receiver demands to receive a specific subset of the sources.

[Rasala, Lehman and Harvey] ~2004

There exist networks where using network coding we can transmit to each receiver rate equal to its min-cut, while without network coding we lose a factor of 1/h .

Different min-cuts
Specific Demand Networks

Each receiver demands to receive a specific subset of the sources.

[Dhougherty, Freiling and Zeger] ~2004

There exist networks that are solvable using nonlinear operations, and not solvable otherwise.

Different min-cuts

Non-uniform Demand Network

Each receiver demands to receive rate equal to its min-cut.

[Cassuto and Bruck] ~2005

Different min-cuts
Non-uniform Demand Network

Each receiver demands to receive rate equal to its min-cut.
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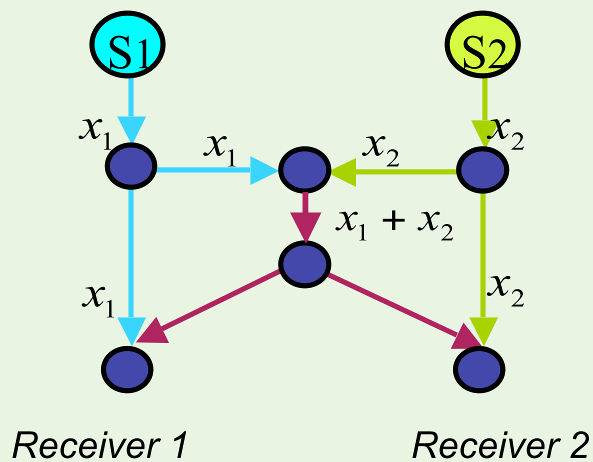
- There exist non-uniform demand networks that are not solvable.



Different min-cuts
Non-uniform Demand Network

Each receiver demands to receive rate equal to its min-cut.
[Cassuto and Bruck] ~2005

- There exist non-uniform demand networks that are not solvable.
- It is NP-complete to decide whether a non-uniform demand network is solvable.

Non-uniform Demand Network
(a,ß) Relaxation

Consider a network represented as a graph with unit-capacity edges, h unit-rate information sources S_1, …, S_h co-located on the same root node and K receiver groups $R_1,\ldots,R_K$.

Group R_i contains N_i receivers, and the min-cut to each receiver in R_i is h_i.

$$h \geq h_1 > h_2 > \cdots > h_K$$

Non-uniform Demand Network
(a,ß) Relaxation

Characterize space of achievable pairs (a,ß)

We say that a pair (a,ß) is achievable if there is a transmission mechanism (network and channel coding, routing, time-sharing) such that some $ßN_i$ of the receivers in group R_i receive rate ah_i.

Non-uniform Demand Network
(a,ß) Relaxation

Characterize space of achievable pairs (a,ß)

Is the pair (a,ß) achievable for constant a and ß?

Non-uniform Demand Network (1,1)

- For K=1, the pair (1,1) is achievable

  (main theorem in network coding, Li and Li result)

- For K>1, the pair (1,1) is not achievable

  (Cassuto and Bruck result)

Non-uniform Demand Network (1,ß)

We want some $ßN_i$ of the receivers in group R_i receive rate equal to their min-cut h_i

Is the pair (1,ß) achievable for constant ß?

Non-uniform Demand Network (1,ß)

We want some $ßN_i$ of the receivers in group R_i receive rate equal to their min-cut h_i

Is the pair (1,ß) achievable for constant ß?

NO

Non-uniform Demand Network (1,ß)

We want some $ßN_i$ of the receivers in group R_i receive rate equal to their min-cut h_i

Is the pair (1,ß) achievable for constant ß?

NO

Proposition 1 There are networks with K=2 receiver groups and 2N receivers for which no point (1,ß) with ß>2/N is achievable.

Non-uniform Demand Network (a,1)

We want all N_i of the receivers in group R_i to receive rate ah_i

Is the pair (a,1) achievable for constant a?

Conjecture: Generally NO, but…

(a, 1)

Undirected Graphs

Theorem 1

In undirected graph instances a>1/2 is achievable even if only routing is allowed

Generalizes the Li and Li result to non-uniform demand networks

$(a,1)$

Theorem 1

Elements of the proof:

1. Convert the undirected graph to a directed graph

2. Decomposition theorem

(Bang-Jensen,Frank, Jackson 1995)

Consider a directed graph, and N receivers having different min-cuts from a common source vertex. If the in-degree of each vertex is larger or equal to the out-degree, there exist edge-disjoint partial Steiner trees such that each receiver appears a time equal to its min-cut as a leaf of these trees.

(a, 1)
Directed Graphs

$$h_1 > h_2 > \cdots > h_k$$

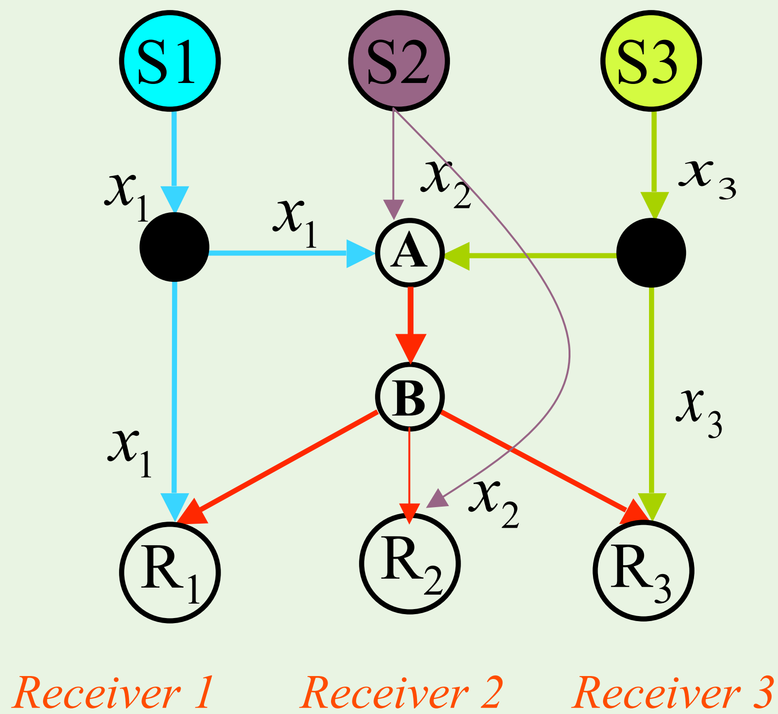
Proposition 2 (simple achievable strategies)

- $a \geq \frac{h_k}{h_1}$ *transmit to the group with the smallest min-cut*
- $a \geq \frac{1}{K}$ *perform time-sharing*
- $a \geq \frac{1}{k - \sum_{i=1}^{k-1} \frac{h_{i+1}}{h_i}} \geq \frac{1}{m \log \frac{h_1}{h_k}}$

Question

Is the pair (a,1) achievable for constant “a” over directed graphs?

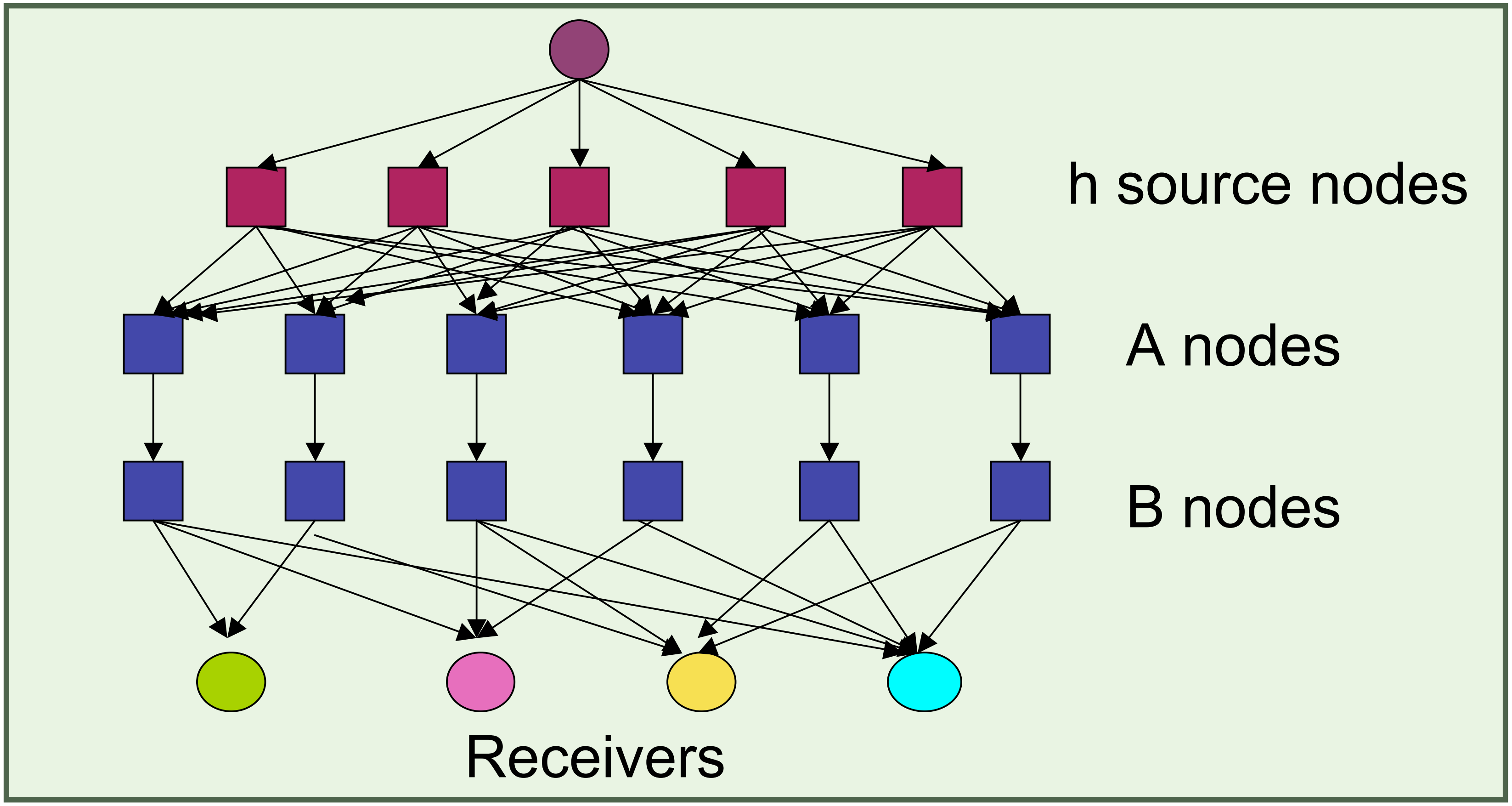
Effect of Source co-location



Effect of Time-Slots and Network Size

If we are restricted to convey the information rate within L time-slots and L is bounded, we can construct networks large enough so that
(1,a) is achievable only for “a” arbitrarily small.

Effect of Time-Slots and Network Size



Effect of memory

The achievable information rate can be substantially impacted by allowing intermediate nodes to store information that they receive and forward it selectively in future time slots.

Open Question

Is the pair (a,ß) achievable for constant a and ß?