Designing Failure-Tolerant Network Codes

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Outline

Multicast Networks

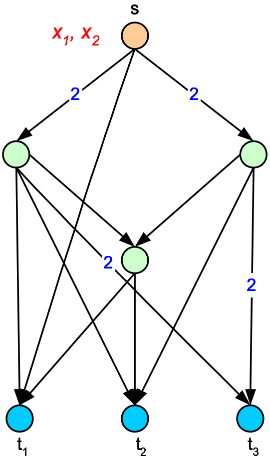


Figure: Example of a multicast network.

- Network represented by a directed graph $G(V, E)$
- Capacity function $c(e) : E \to \mathbb{N}$
- Source s where h packets $x_1, \ldots, x_h$ are available
- k destination nodes $t_1, \ldots, t_k$ that require all the packets
- Edges are susceptible to failure

The Problem

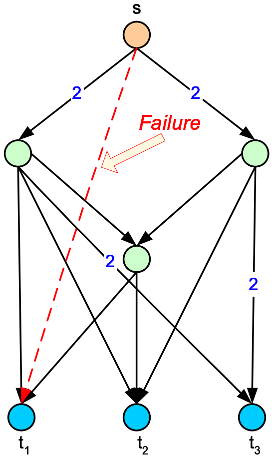


Figure: Failed edge in a network

- A failed edge is deleted from the network

Goal

Find a communication scheme that will guaranty the delivery of all the packets to all the destinations in the case of any single edge failure.

Feasibility

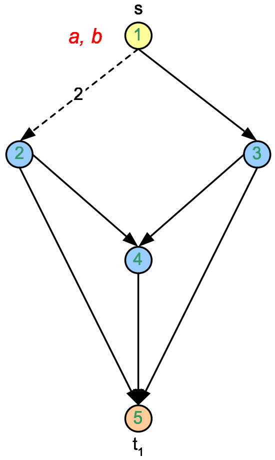


Figure: A non feasible unicast network

Definition (Feasible Network)

A multicast network $\mathbb{N}$ is feasible if there exists a flow of value h from the source to each destination in any subnetwork obtained by deleting a single edge in $\mathbb{N}$.

Network Flows

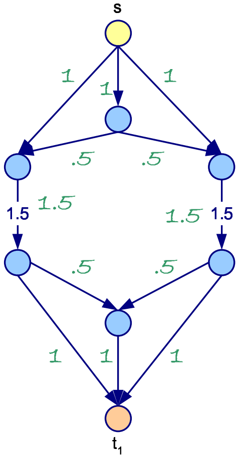


Figure: A flow of value 3 in a network.

Definition (Flow of value h)

A flow $\mathbf{b}$ of value h from s to t in a network $\mathbb{N}$ is an assignment of a real number $b(e)$ to each edge $e \in \mathbb{N}$ such that

1. $b(e) \leq c(e)$
2. flow out at $s = h$
3. flow in at $t = h$
4. flow in = flow out at intermediate nodes

Example: Rerouting

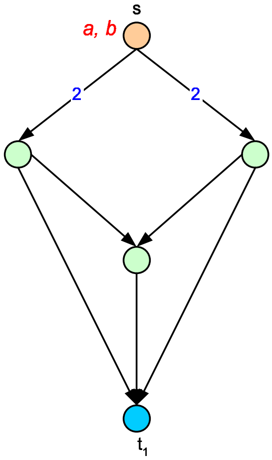
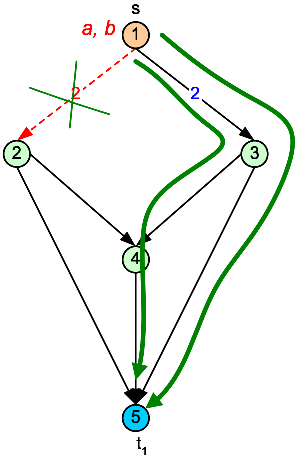


Figure: A feasible unicast network.

Definition

A unicast network is a multicast network with a single destination.

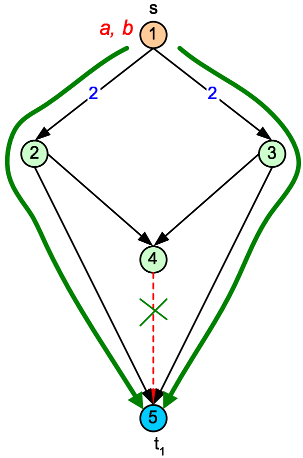
Example: Rerouting



Rerouting

For every edge $e \in E$, find a flow of value h in the subnetwork $G \setminus e$, that will protect against the failure of e.

Example: Rerouting



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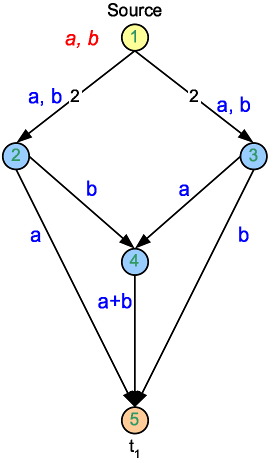
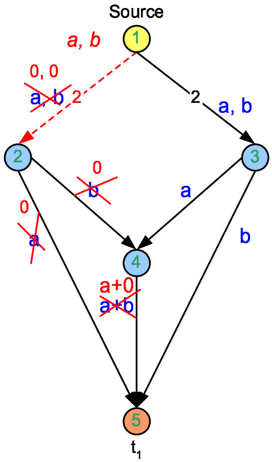
Better than Rerouting



- a and b are bits
- "+" is the bit xor operation
- the packet carried by a failed edge is always zero
- A kind of "coding" is done at node v_4

Better than Rerouting

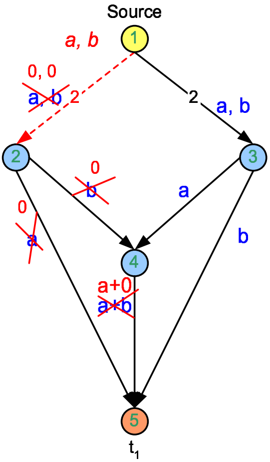


- Single edge failure
- The destination will always be able to decode the original packets a and b

failure of	m_{25}	m_{45}	m_{35}
ϕ	a	$a+b$	b
(v_1, v_2)	0	a	b
(v_1, v_3)	a	b	0
(v_2, v_4)	a	a	b
(v_3, v_4)	a	b	b
(v_2, v_5)	0	$a+b$	b
(v_3, v_5)	a	0	b
(v_4, v_5)	a	$a+b$	0

- Instantaneous recovery!

Better than Rerouting



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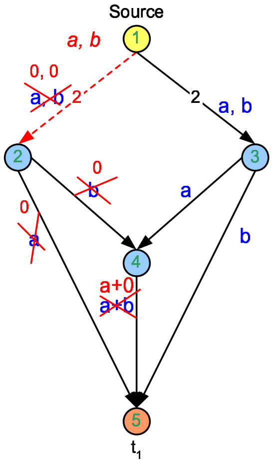

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Linear Network Codes

Network Coding

- The packet carried by an edge $e(u, v)$ is
  - A function of the original packets if u is the source node
  - Otherwise, a function of the packets carried by edges incoming to u
- The set of all the edge functions is called a network code

Linear Network Codes

- Packets at the source belong to some finite field $GF(q)$
- The edge functions are linear over that field

Robust Network Codes

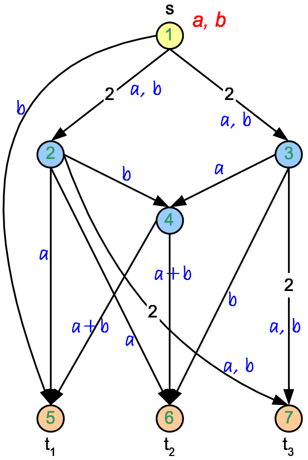


Figure: Example of a robust network code

Definition

A robust network code, for a multicast network, is a linear network code that, in the case of *a single edge failure* will guaranty

- the delivery of all the packets
- to all the destinations

Related Work

- Koetter et al. (03) showed that linear robust network codes always exist for feasible multicast networks.
- Jaggi et al. (04)
  - designed a polynomial time algorithm for finding robust network codes
  - gave an upper bound on the minimum field size over which such codes exist for a given multicast network, that is fk
    - f is the number of failure patterns. Here, $f = |E|$
    - k number of destinations

Summary of the Results

1. We focus on feasible unicast networks with $h = 2$
2. We show that such networks have a very specific structure. They can be constructed by the concatenation of three blocks that we describe
3. We prove, constructively, that robust network codes exist for these networks over $GF(2)$
4. We show that for multicast networks with k destinations and $h = 2$, a field of size larger than $5k$ is sufficient for finding robust network codes

Minimal Networks

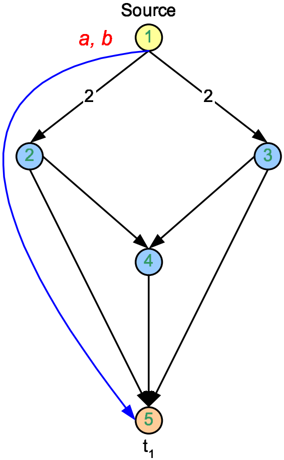
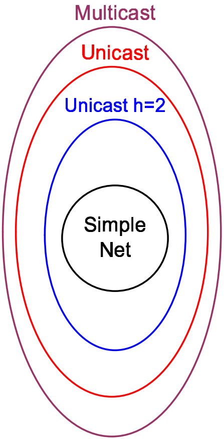


Figure: A non minimal unicast network

Definition

A multicast network is minimal if all its *subnetworks* , obtained by deleting an edge or reducing its capacity, are *not feasible*.

Simple Networks



Definition

A unicast network ($h = 2$) is a simple network iff it is

- feasible
- minimal
- All of its nodes are of degree 3

Reduction to Simple Networks

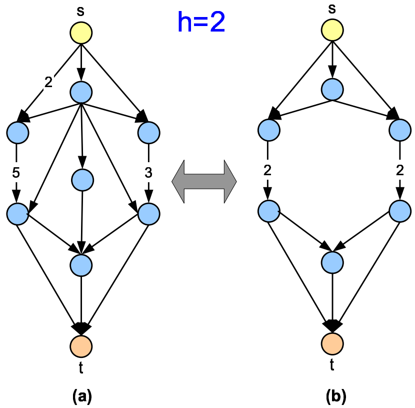


Figure: (a) unicast net. (b) corresponding simple net.

Theorem

Let $\mathbb{N}$ be a feasible unicast network ($h = 2$). Then, there exists a simple network $\mathbb{N}'$ such that if $\mathbb{N}'$ has a robust network code over $GF(q)$, then $\mathbb{N}$ has also one over the same field.

Flow Network

- We transform the problem of protecting against edge failures in a network $\mathbb{N}$, into the problem of studying the properties of flows in a corresponding network $\bar{\mathbb{N}}$

Definition (Flow Network)

To each unicast network $\mathbb{N}$ with $h = 2$, we associate a flow network $\bar{\mathbb{N}}$ defined on the same graph but where edge capacity are reduced from 2 to 1.5, and all the other capacities are kept the same.

Example of a flow network

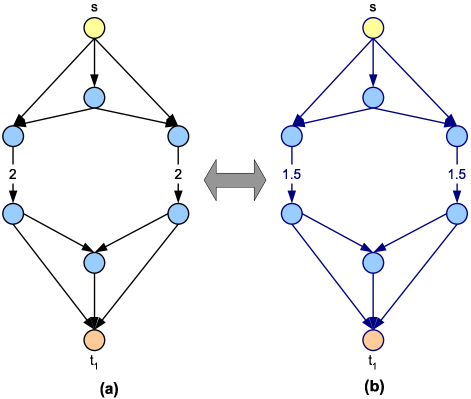


Figure: (a) feasible unicast net $\mathbb{N}$. (b) Corresponding flow net $\bar{\mathbb{N}}$.

A Property of Flow Networks

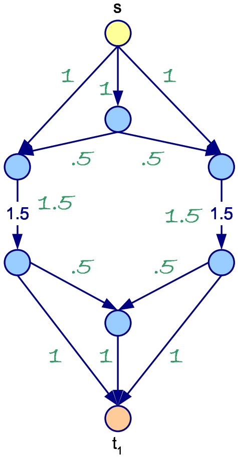


Figure: A flow of value 3 in a flow network.

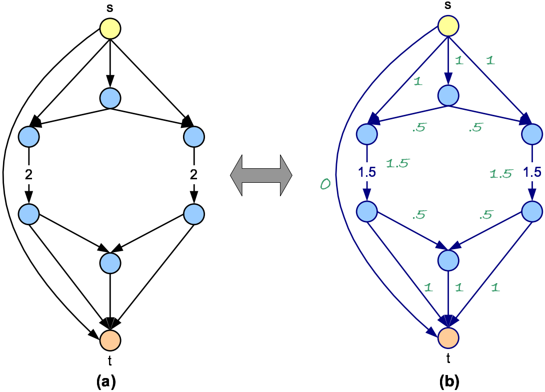
Theorem

A unicast network is feasible if and only if the corresponding flow network admits a flow of value 3 from s to t.

Flows and Minimality

Theorem

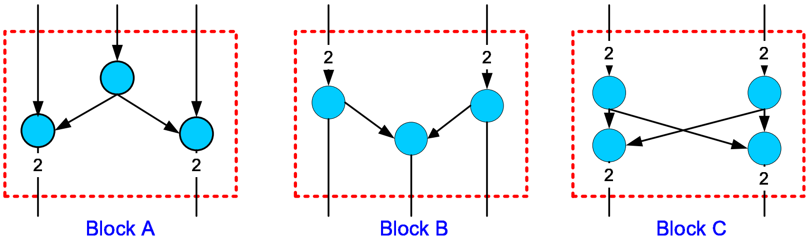
Let $\mathbb{N}$ a simple (minimal& feasible) network. Then, in the corresponding flow network $\bar{\mathbb{N}}$, all flows of value 3 are nowhere-zero flows.



Structure of Simple Networks

Theorem

All simple networks $\mathbb{N}$ can be decomposed into the blocks A, B and C depicted below.



Example

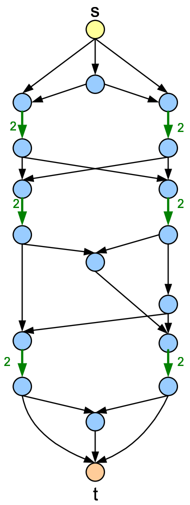


Figure: A simple network

Example

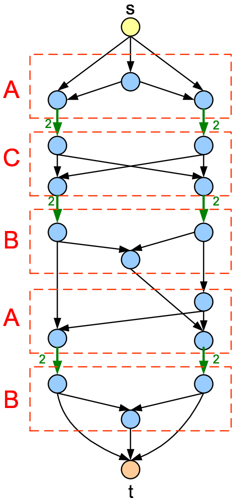
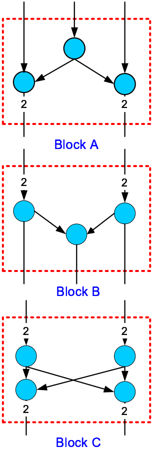


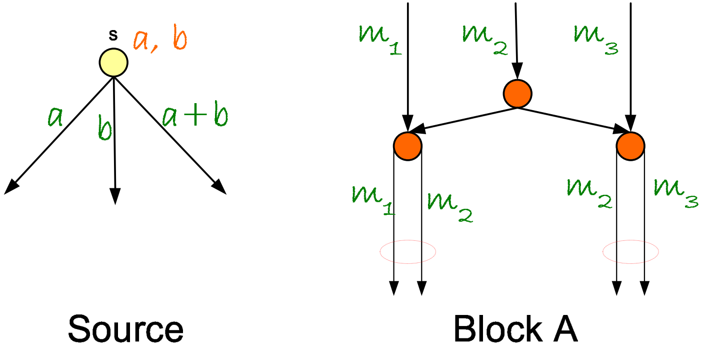
Figure: Block decomposition of a simple network

Sketch of Proof

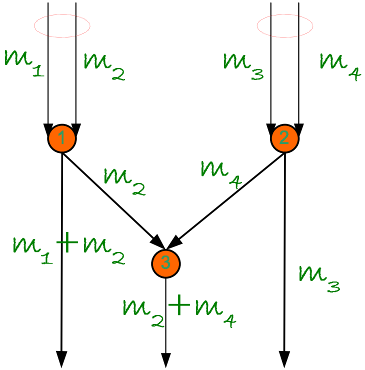


- The proof of the *block decomposition theorem* of simple networks is based on
  - Residual networks
  - The augmenting cycle theorem
- Any configuration, other than blocks A, B and C
  - will result in a flow with some edge carrying a *zero* flow.
  - contradicts the minimality of the simple network

Robust Network Code

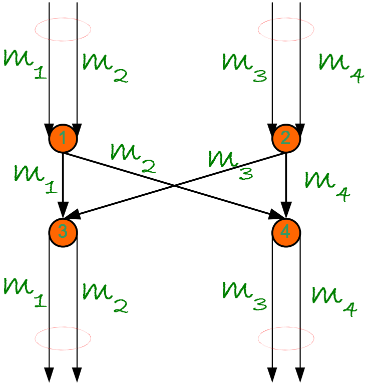


Robust Network Code



Block B

Robust Network Code



Block C

Proof of Robustness

- A simple network always ends by a block B
- The proof of the robustness of the previous network code
  - is done by induction on the number of blocks B in the network
  - shows that the output of any block B is always a subset of at least two elements of the set $\{a, b, a + b\}$

Beyond Two Packets

- The flow network technique does not generalize for $h > 2$
- Finding the structure of feasible and minimal networks seems hard for large values of h

First Conjecture

Conjecture 1

There exists a function $f(h)$, such that, for all prime powers $q \geq f(h)$, there exist robust network codes over $GF(q)$ for all unicast networks with h packets.

- $f(h)$ does not depend on the network
- $f(2) \geq 2$

Beyond Unicast

Theorem

Consider a multicast network $\mathbb{N}$ with $h = 2$ packets and k destinations. Then, there exists a robust network code for $\mathbb{N}$ over $GF(q)$ for all $q \geq 5k$.

Lemma

If m flows are needed to protect against all single edge failures in $\mathbb{N}$, then there exists a robust network code $\mathbb{N}$ over $GF(q)$ for all $q \geq m$ (Jaggi et al. 04).

Beyond Unicast

Theorem

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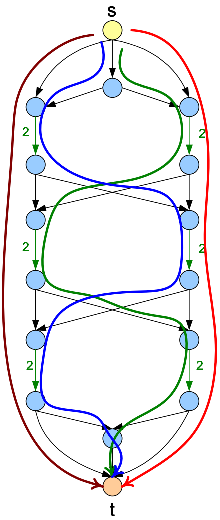
Proof



- For a unicast network, at most 5 flows are sufficient for protecting against failures
  - Red-Maroon for edges in the middle
  - Green-Red and Blue-Red for edges on the left
  - Blue-Maroon and Green-Maroon for edges on the right
- If we repeat this for each destination, we get the $5k$ bound

Second Conjecture

Conjecture 2

There exists a function $g(h)$, such that, for all prime powers $q \geq kg(h)$, there exist robust network codes over $GF(q)$ for all multicast networks with h packets and k destinations.

Summary

1. We addressed the problem of constructing robust network codes for multicast networks
   - focused on unicast networks with $h = 2$ packets
2. We described the structure of feasible and minimal unicast networks ($h = 2$)
   - It can be constructed by the concatenation of three blocks
3. Constructed a robust network code for these networks
   - permits instantaneous recovery from any single edge failure
   - over $GF(2)$
4. Showed that a field of size q, where $q \geq 5k$, is always sufficient for finding robust codes for multicast networks with 2 packets and k destinations