Toward Network Coding for Interference Networks

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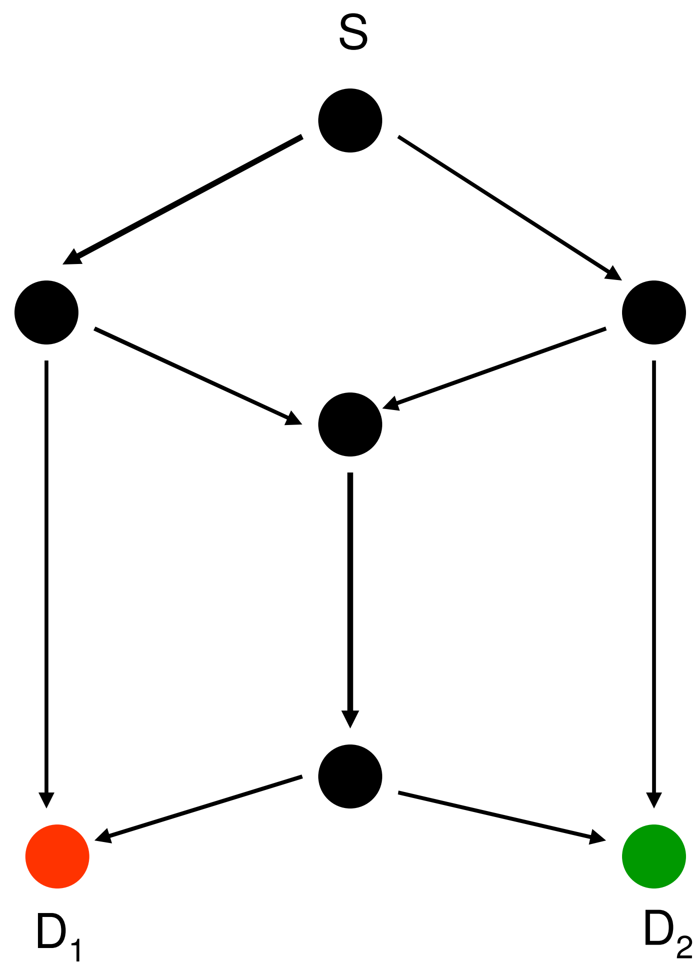
Outline

- Introduction
  - Network coding
  - Wireless models – broadcast erasure networks
- Our system model
  - Finite-field operations
  - Both broadcast and interference constraints
- Upper bound
- Network coding strategy
  - Achieves rates asymptotically close to u.b.
- Capacity gains due to fading

Network Coding – Wireline Networks

- All links at rate 1
- Single-source multicast



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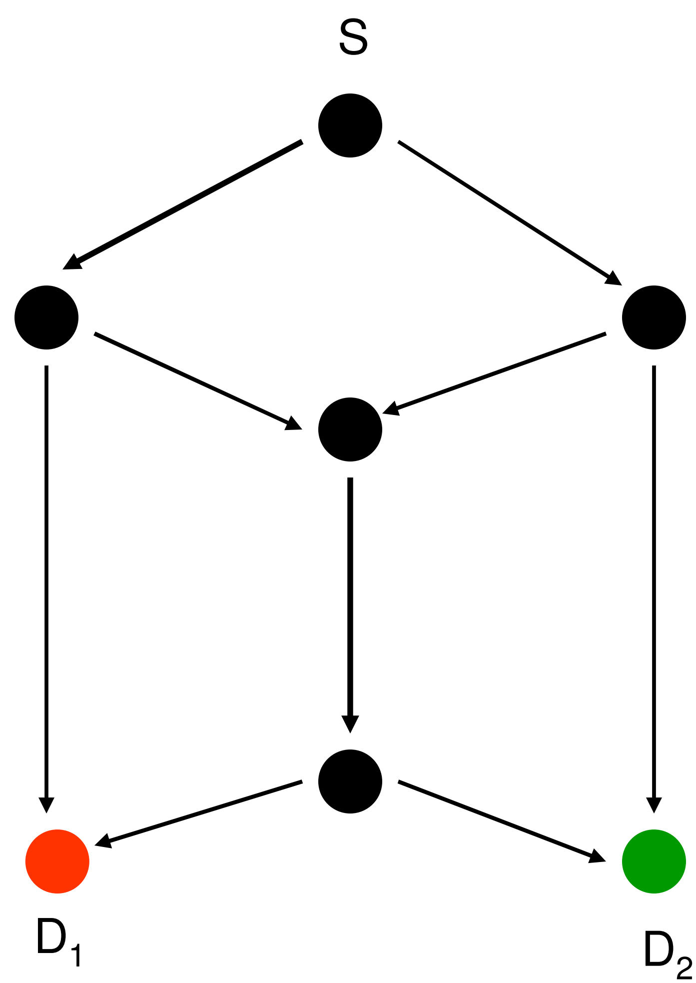
  - Same as min-cut



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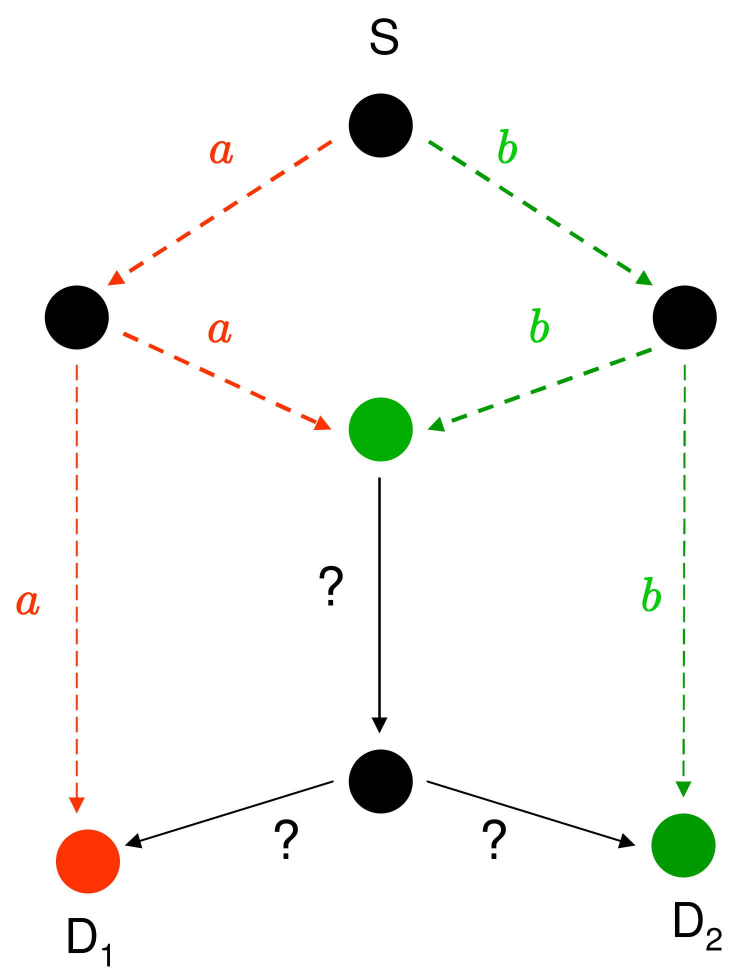

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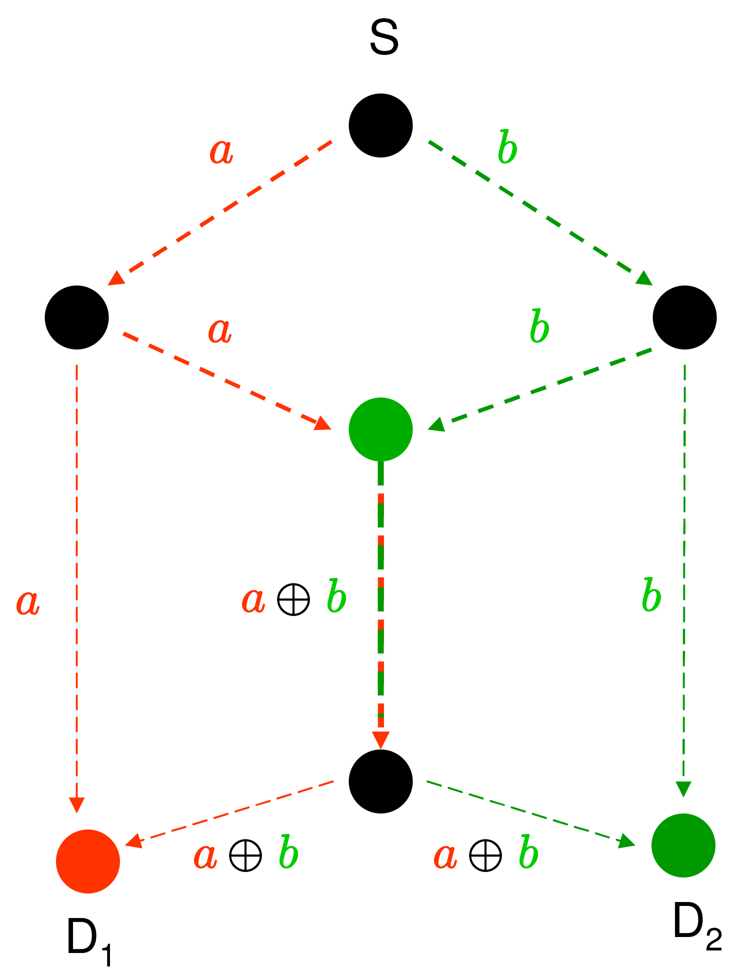
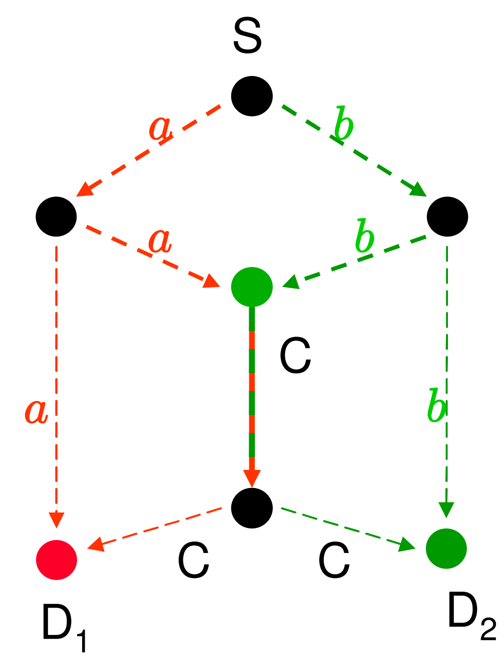
- Network coding can
  - By coding at intermediate node



Network Coding – General Wireline Networks

Theorem [AhCaLiYe00]: Multicast capacity = $\min_i$ {min-cut for D_i }

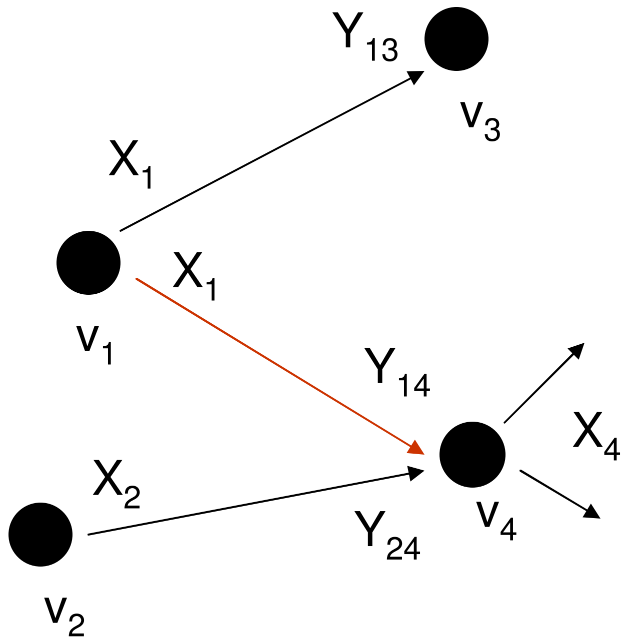
- **Upper bound**
  - Min-cut bound for each D_i
- **Achievability** [Ho et al 03]
  - Source sends messages from F_q
  - Nodes perform Random Linear Coding (RLC) over received messages: $C = \alpha_1 a + \alpha_2 b$, $\alpha_i \in F_q$
  - D_1 decodes source messages from received vectors: $Y_1=(Y_{11}=a, Y_{12}=C)$
  - Achieves rates arbitrarily close to min-cut bound for sufficiently large q



$$Y_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Broadcast Erasure Networks (BEN)

- Directed graph G=(V,E)
- Each link $e \in E$ is independent erasure channel
  - $P(Y_{14} \mid X_1, X_2) = P(Y_{14} \mid X_1)$



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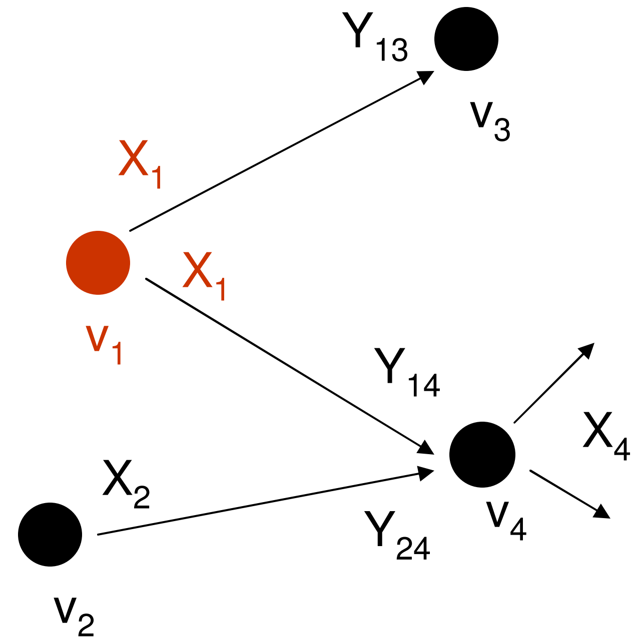

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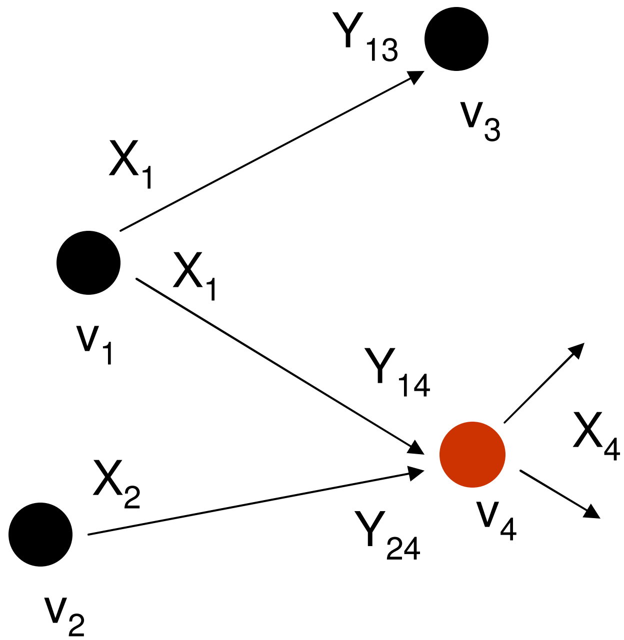
  - $X_4 = f(Y_{14}, Y_{24})$



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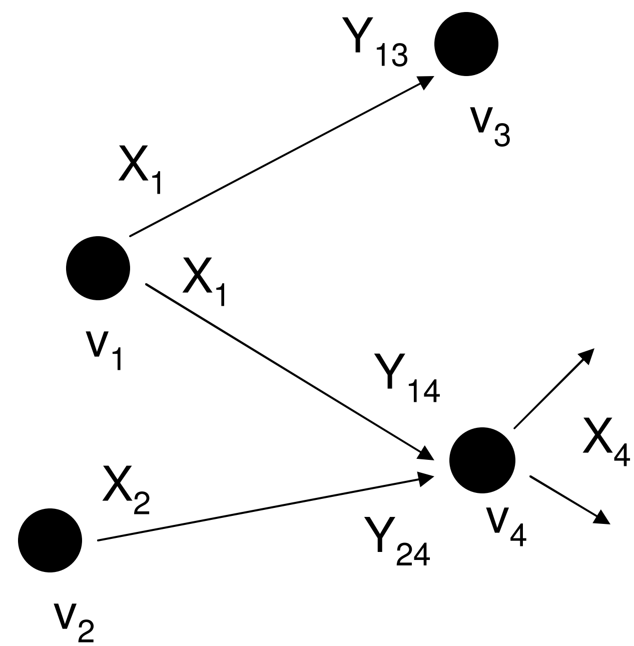
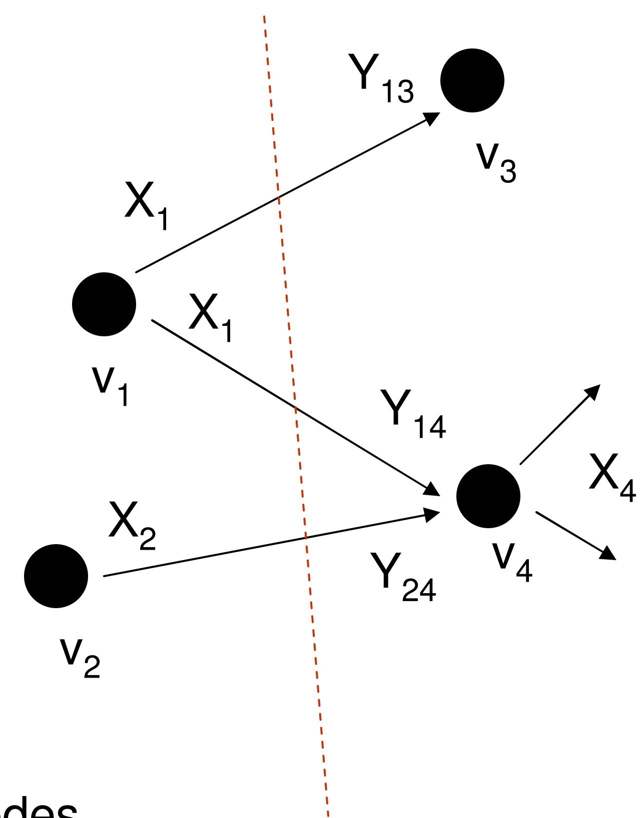


Does not model interference

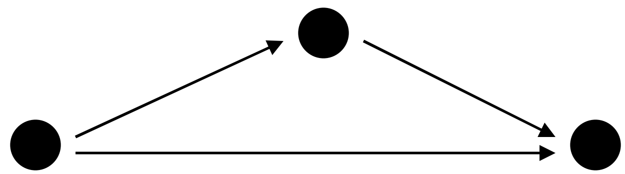
Broadcast Erasure Networks -- Capacity

- Results for directed acyclic graphs
- Theorem [DanGow04, LunMed04]:
  Capacity = $\min_i$ {*generalized* min-cut for D_i}
  - e.g., $(1 - \varepsilon_{13}\varepsilon_{14}) + (1 - \varepsilon_{24})$
- Upper bound
  - Follows from min-cut bound
    $\min_{cut} I(X_{cut\text{-}l}; Y_{cut\text{-}r} \mid X_{cut\text{-}r})$
- Achievability
  - [DanGow04] Random coding at nodes
    - need to keep track of erasure patterns
  - [LunMed04] model as Hypergraph, RLC at nodes
    - track flow of innovative packets
    - generalized to arbitrary arrival processes and correlated erasure patterns

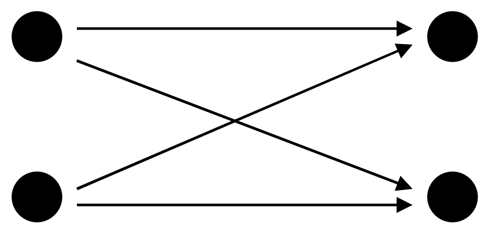


Wireless Broadcast and Interference Networks (WBAIN)

- Above and other results model broadcast but not interference
- Interference is challenging to analyze
- Capacity region not known for even simple network configurations
  - Single-relay channel



  - Interference channel



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WBAIN – A Finite-Field Model

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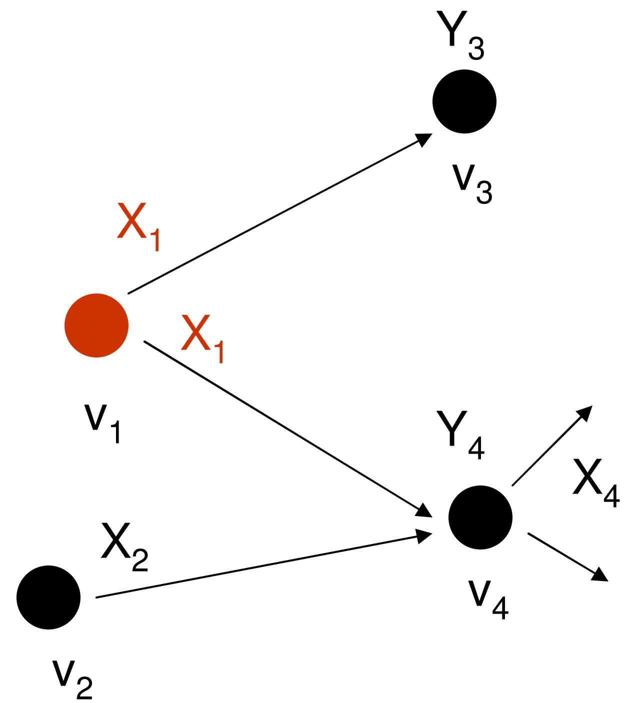

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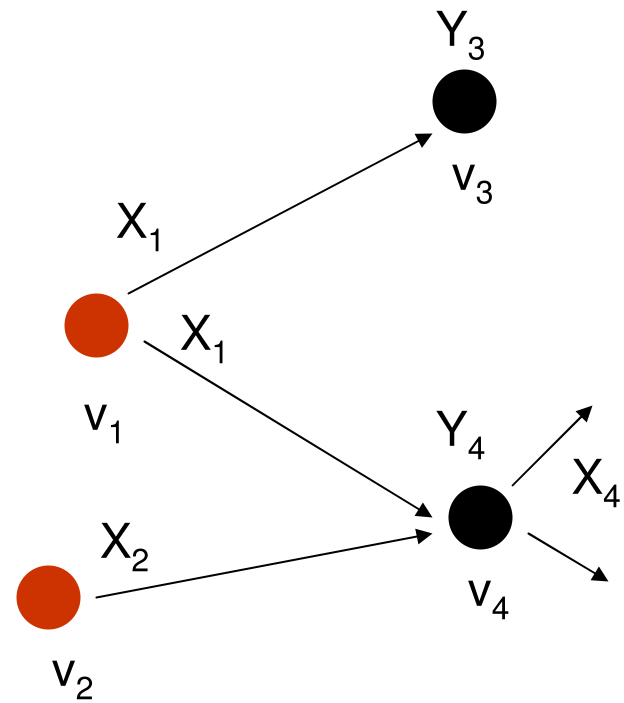
- Model power constraint by rate
  - v_i can send at rate $\leq R_i$



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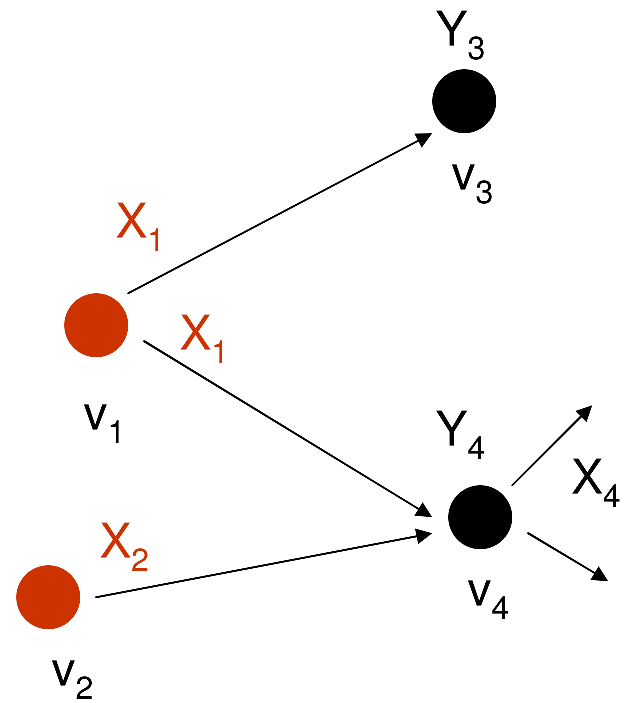
- All operations over finite field F_q
  - Each node transmits vectors from F_q
  - $\log q \geq \max_i R_i$



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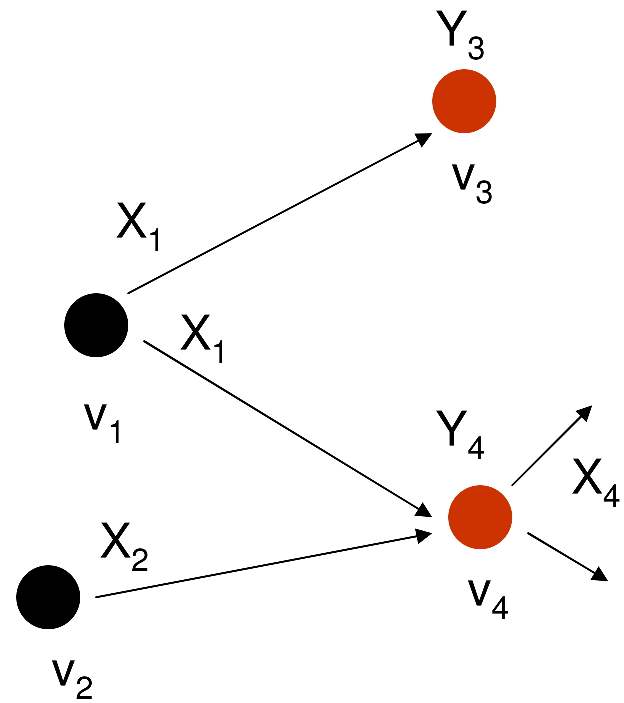
- Two reception models
  - With or without fading

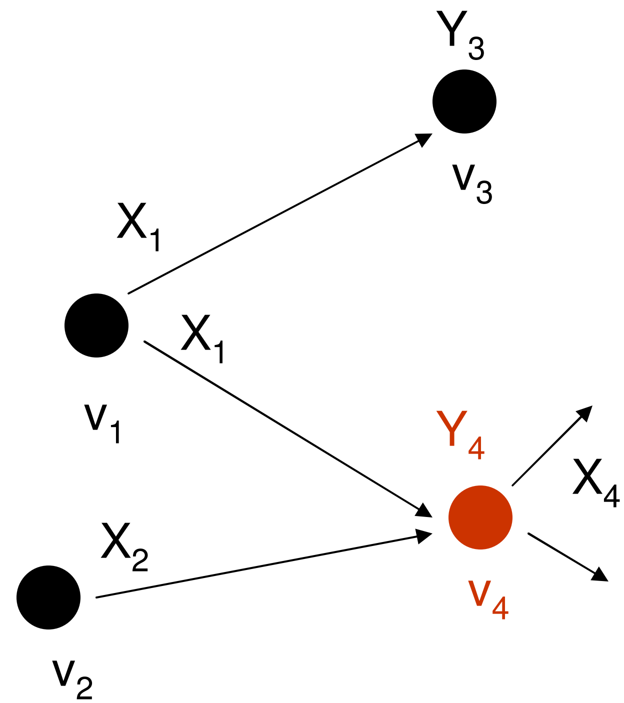


WBAIN – A Finite-Field Model (contd.)

- Non-fading model:

  v_4 receives Y_4, where Y_4

  - $= X_1 + X_2 \in F_q$ with prob. $1-\varepsilon_4$
  - $= \phi$ (erasure) with prob. ε_4
  - Erasures are independent across receivers



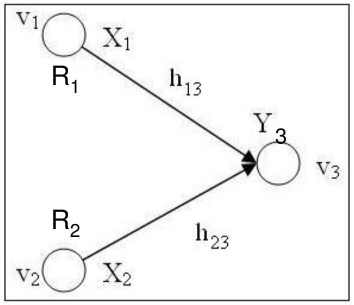
- Fading model:

  As above, except when no erasure

  - $Y_4 = h_{14} X_1 + h_{24} X_2$
  - h_{ij} uniform i.i.d. over F_q

Upper Bound on WBAIN with Fading

- Bound on capacity of finite-field MAC



$$C_3(q) = \max_{\{H(X_j) \le R_j\}} I(Y_3; X_1, X_2 | h_{13}, h_{23})$$

$$= \max_{\{H(X_j) \le R_j\}} H(Y_3 | h_{13}, h_{23}) - H(Y_3 | X_1, X_2, h_{13}, h_{23})$$

$$= \max_{\{H(X_j) \le R_j\}} H(h_{13}X_1 + h_{23}X_2 | h_{13}, h_{23}) - 0$$

$$= q^{-2}((q-1)(R_1 + R_2) + (q-1)^2 \min\{R_1+R_2, \log q\})$$

$$\le \min\{(1-q^{-1})(R_1 + R_2), (1-q^{-2}) \log q\}$$

Upper Bound on WBAIN with Fading

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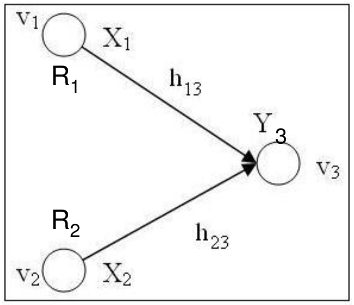

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- Node i receives transmissions from nodes in J

$$C_i(q) \le \min \{(1-q^{-1})(\textstyle\sum_{j\in J} R_j), (1-q^{-\delta_I(i)}) \log q\}$$

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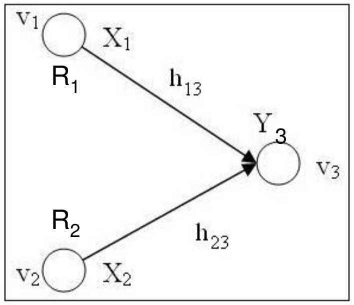

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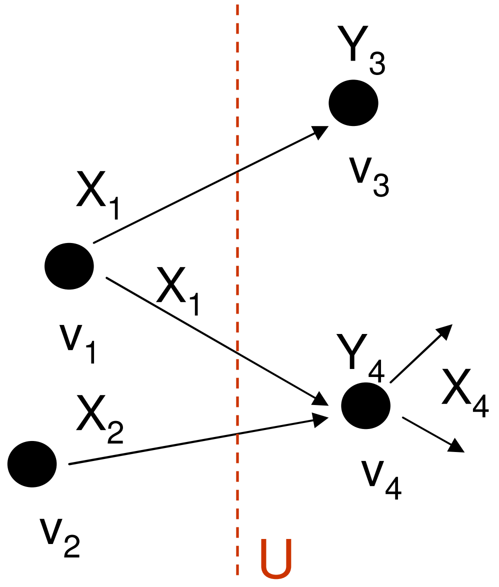
- More generally, total rate across cut U bounded by

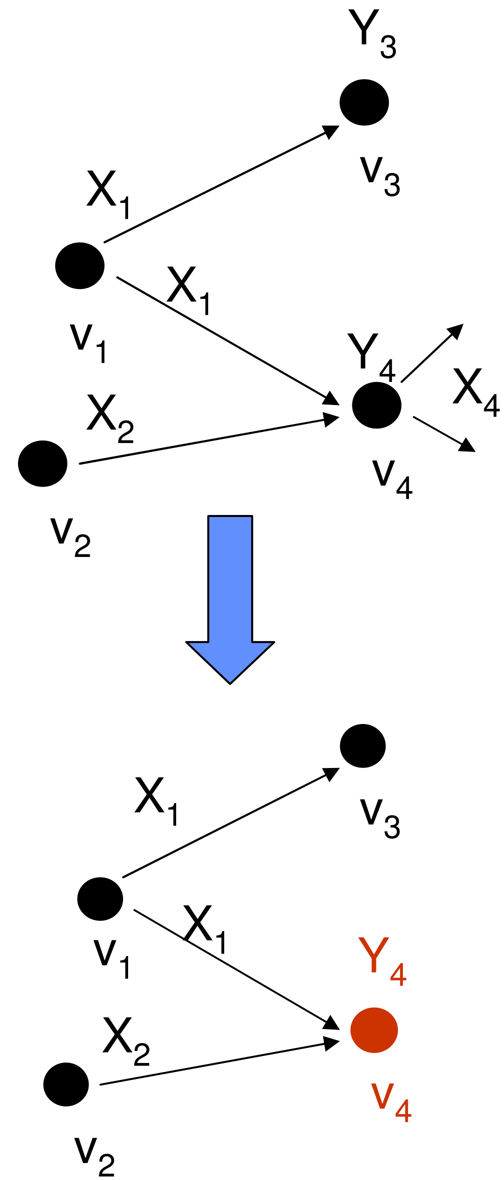
$$C_U(q) \le \max_{\{H(X_j) \le R_j\}} I(Y_3, Y_4; X_1, X_2 | H_{1,2;3,4})$$

$$\le \min \{\textstyle\sum_j (1-q^{-\delta_O(j)}) R_j, \log q \sum_i 1-q^{-\delta_I(i)}\}$$



Upper Bound (contd.)

- Consider Broadcast Erasure Network T(G) having same topology and rates as G, with
  - no interference, e.g., Y_4 receives (X_1, X_2)



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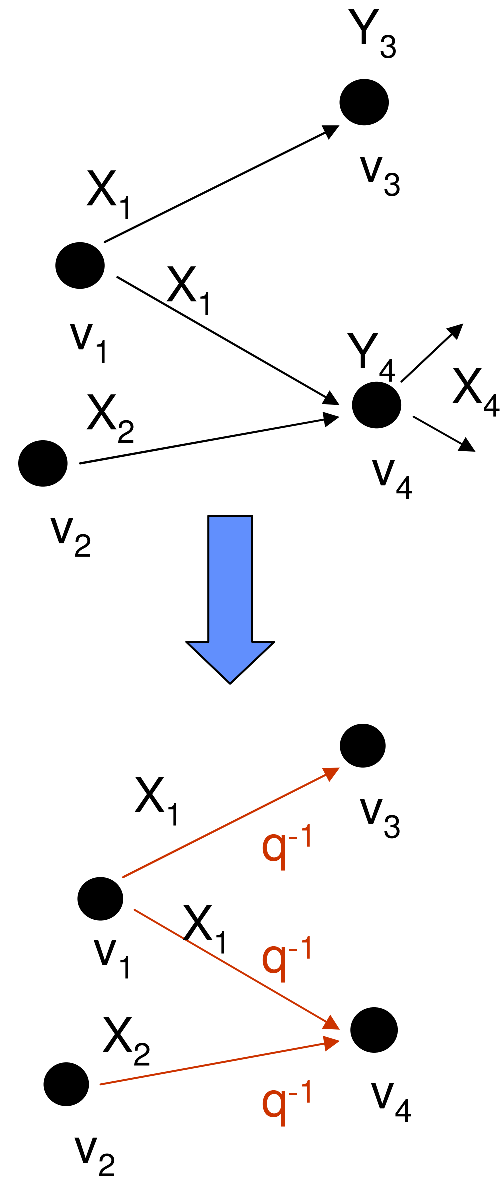

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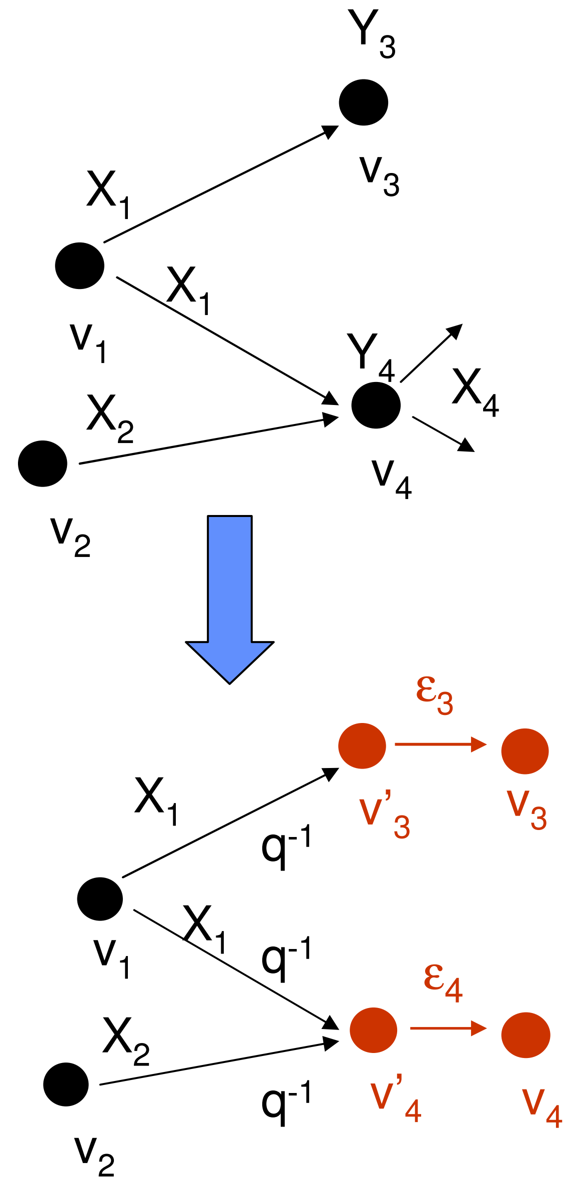

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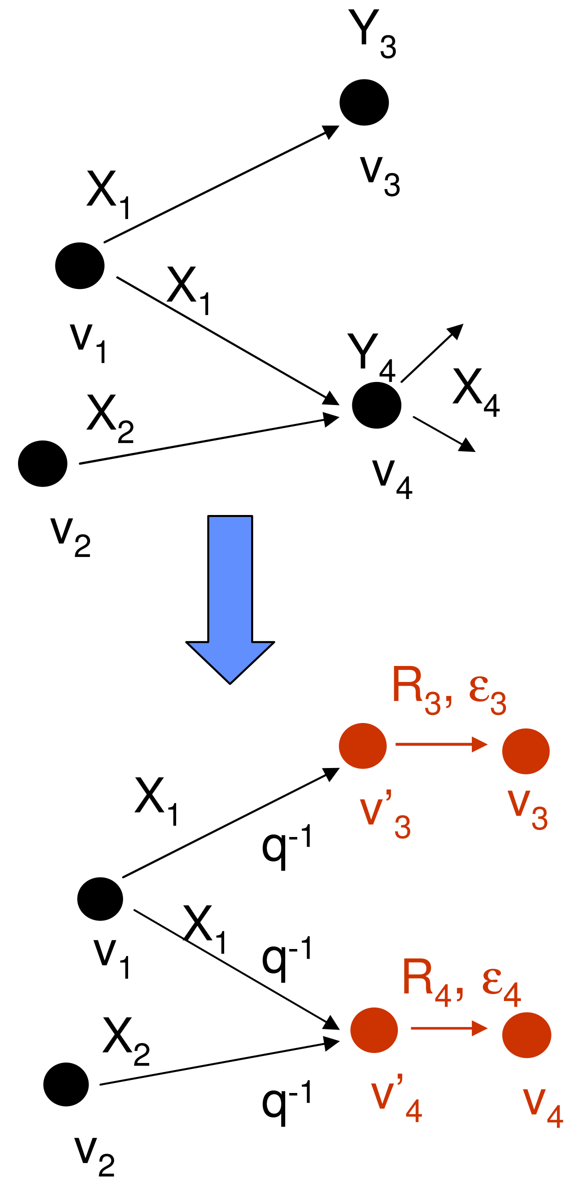

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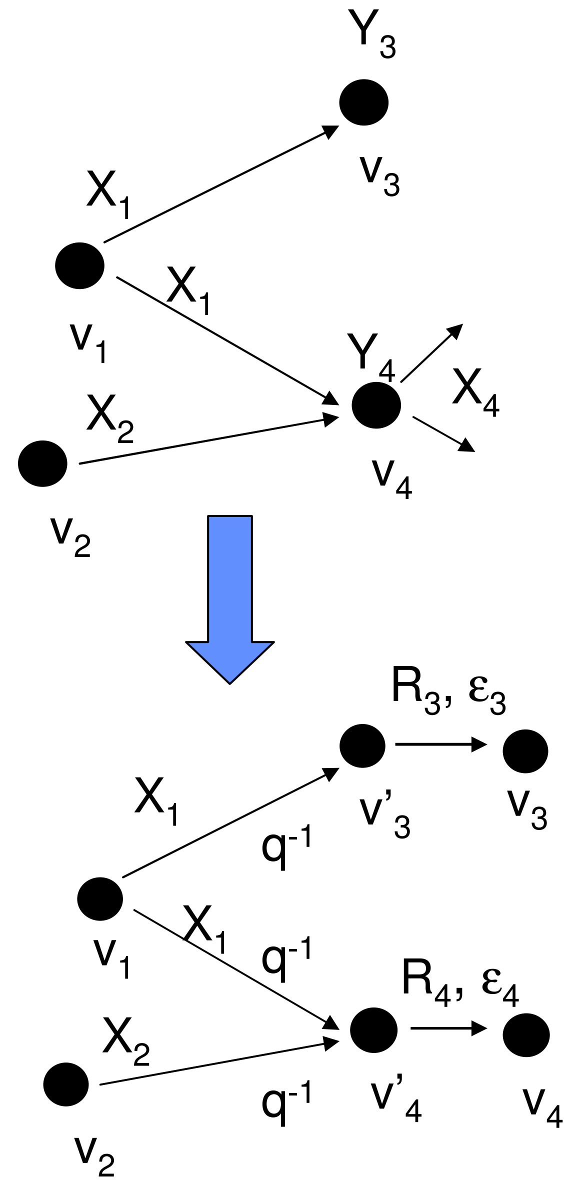
- Theorem: Capacity of WBAIN G over F_q, C_q

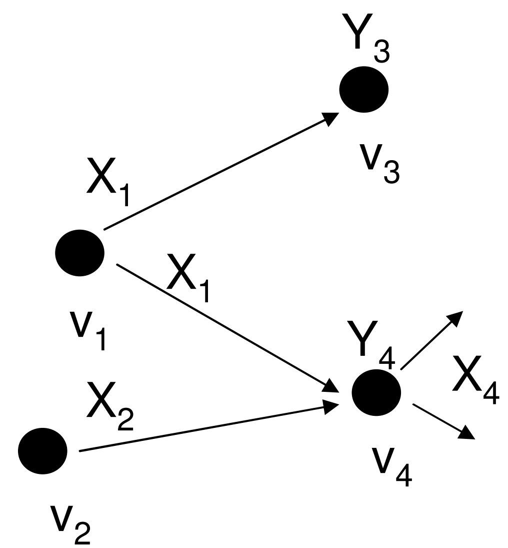
  $\leq$ Capacity of BEN, T(G), $C_s(q)$

  = $\min_i$ {generalized min-cut for D_i}



Coding Strategy for WBAIN with Fading

- For any $\delta>0$, there exists flow vector $\{f_p\}$ for all paths $\{p\}$ between s-d in BEN T(G) such that
  - $\sum_p f_p = C_s \cdot (1-\delta)$
  - $\sum_{p:\, v_i \in p} f_p / (1-\varepsilon_{h(v_i,p)}) \leq R_i \cdot (1-\delta)$
    - $h(v_i,p)$ is next hop from i on path p



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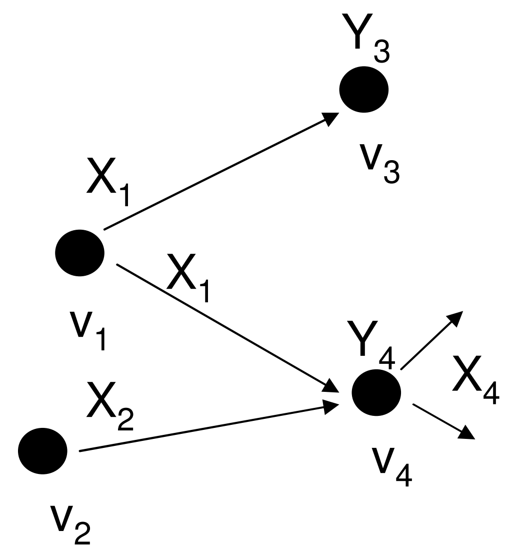

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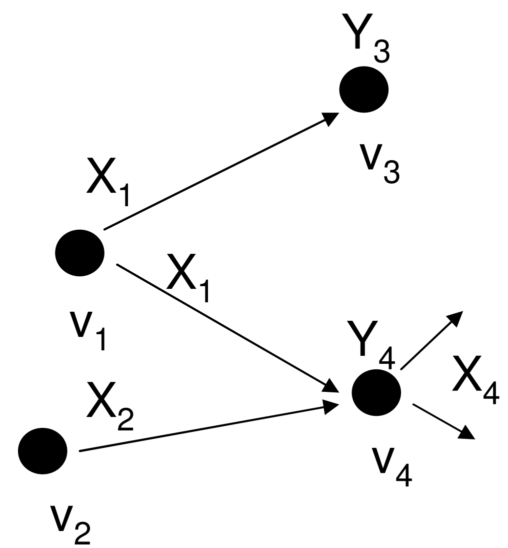

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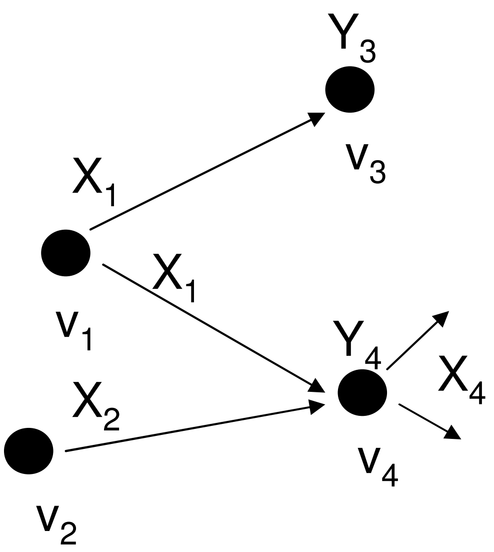

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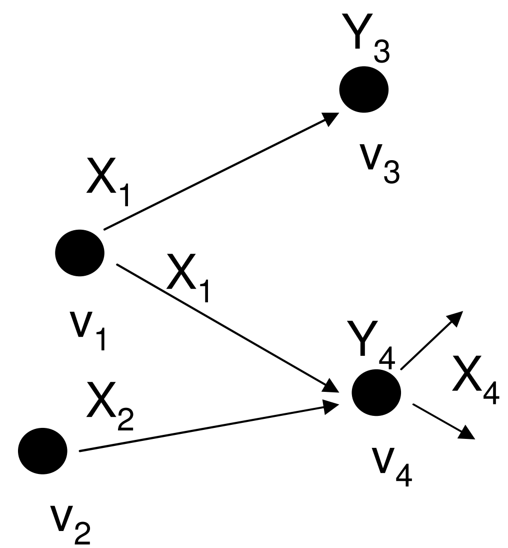
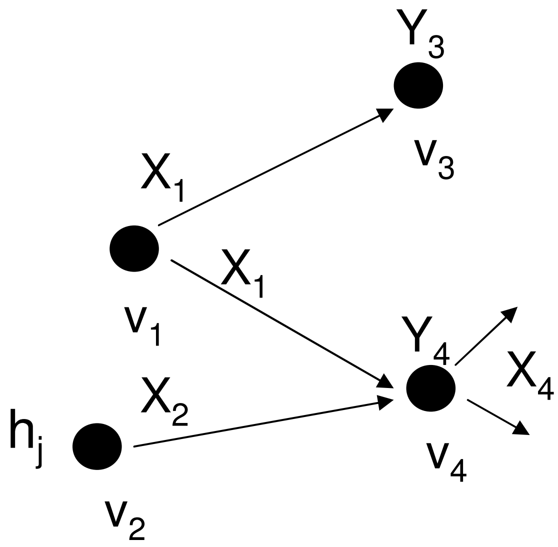


- Theorem: $C_s \cdot (1-O(1/q)) \cdot (1-\delta)$ is achievable in G with uniform i.i.d. fading

Coding Strategy for WBAIN with Fading (contd)

Main steps in achievability proof:

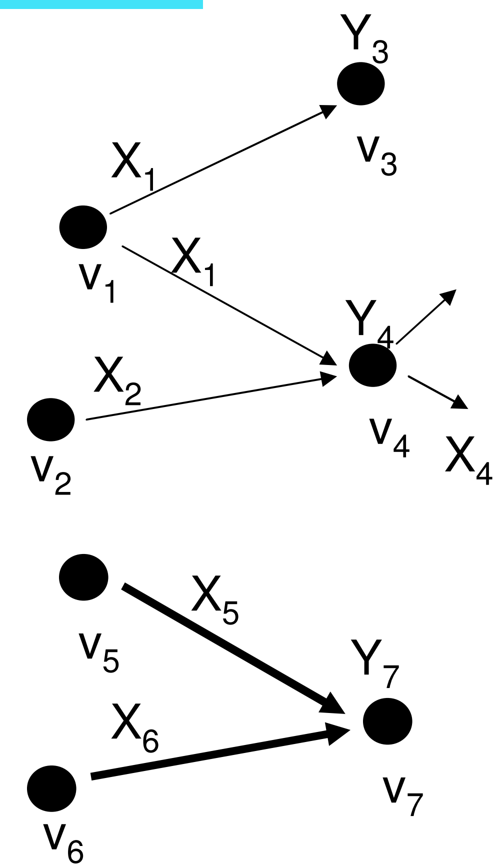
- Track the flow of innovative packets
- Fading helps to maintain innovation rates over different links in a cut
  - in spite of broadcast and interference
- "Bad" fading at node v_j -- $h_j=(h_{ij})_i = 0$ or dependent on $\{h_k\}$ -- reduces rate of innovation by at most $(1-O(1/q))$
- At each hop of path p the rate of innovation is at least $g_p = f_p \cdot (1-O(N_o/q))$
  - N_o = diameter of G
- Achieved rate = $\sum_p g_p = C_s \cdot (1-O(N_o/q)) \cdot (1-\delta)$



Tight bounds on Capacity of WBAIN with fading

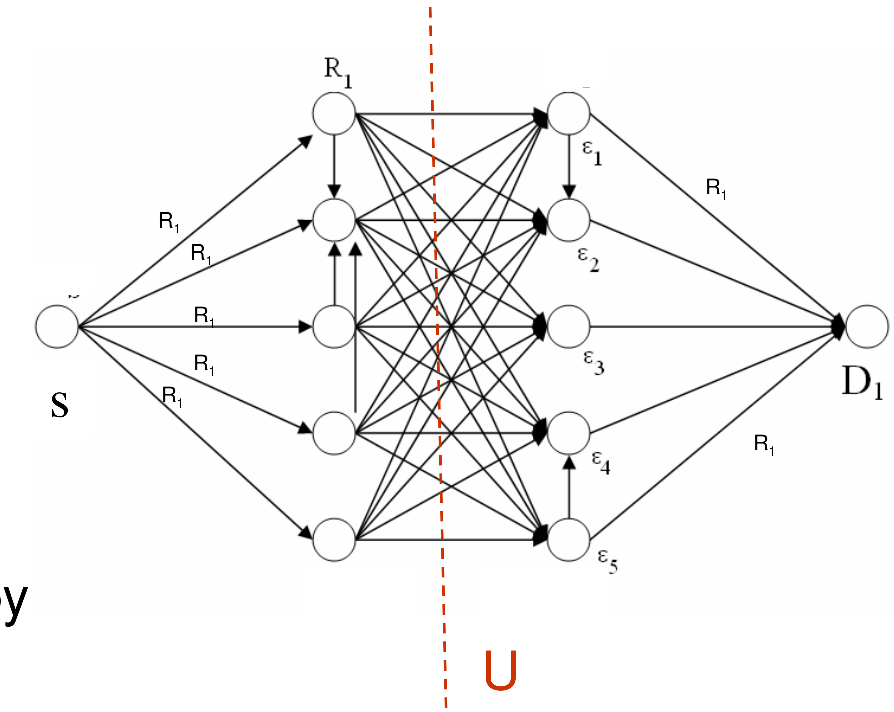
Theorem: $C_s \cdot (1 - O(1/q)) \leq C_q \leq C_s$

- Also holds for heterogenous networks having both wireless and wireline links:
  - Each node can have both types of incoming and outgoing links
  - Node receives weighted sum of vectors sent over incident *wireless* links, $Y_4 = h_{14}X_1 + h_{24}X_2$
  - Node receives separate information over incoming *wireline* links, $Y_7 = (X_5, X_6)$
  - Similarly, when node transmits



Capacity Gains due to Fading – An Example

- Heterogenous network: wireless at cut U, wireline otherwise
- R_1 and q s.t. U is bottleneck cut
  - e.g., $R_1 = \log q$
- Upper bound:
  $C_s \sim \sum_{i=1}^{5} R_1(1-\varepsilon_i) = R_1(5-\sum_i \varepsilon_i)$
- Fading: our strategy achieves
  $C_s \cdot (1-O(1/q))(1-\delta)$
- No fading: capacity is bounded by
  $R_1(1-\prod_i \varepsilon_i)$



- ~5-fold increase in capacity with fading
  - Higher for graphs with larger bottleneck cut

Summary and Future Work

- Finite-field model of interference networks
  - All operations over a finite field
  - Incorporates both broadcast and interference constraints
  - Allows for fading
- Asymptotically tight bounds on capacity for uniform iid fading
  - Upper bound based on results for Broadcast Erasure Networks
  - Achievability through network coding

Some Interesting Issues

- Non-uniform fading?
- Achievable rates under no fading?
- What can we infer about Gaussian channels?
  - Limit of finite-field channels under appropriate distribution remapping?