Broadcasting with an ‘Upgraded’ Relay

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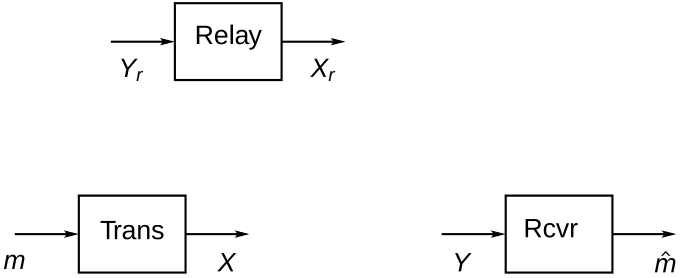
Model

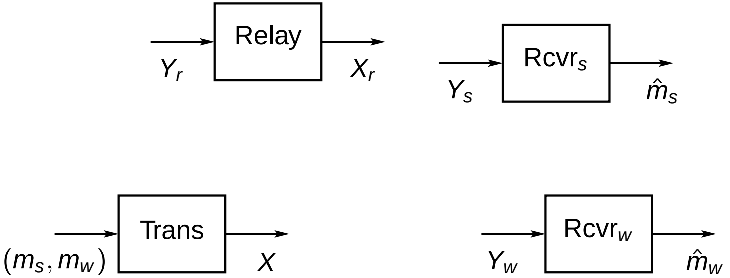


$$Y_r = X + Z_r, \qquad Z_r \sim \mathcal{N}(N_r), \qquad \overline{X^2} \leq P$$

$$Y = X + X_r + Z, \qquad Z \sim \mathcal{N}(N), \qquad X_{r,i} = \phi(Y_r^{i-1}), \ \overline{X_r^2} \leq P_r$$

Noises at time i are independent of the noises before and signals upto and including time i.

Model



$$
\begin{aligned}
Y_r &= X + Z_r, & Z_r &\sim \mathcal{N}(N_r), & \overline{X^2} &\leq P \\
Y_s &= X + X_r + Z_s, & Z_s &\sim \mathcal{N}(N_s), & X_{r,i} &= \phi(Y_r^{i-1}),\ \overline{X_r^2} \leq P_r \\
Y_w &= X + X_r + Z_w, & Z_w &\sim \mathcal{N}(N_w), & N_w &\geq N_s
\end{aligned}
$$

Noises at time i are independent of the noises before and signals upto and including time i.

Encoding — Single Receiver

- Generate $\mathbf{U}(1), \ldots, \mathbf{U}(2^{nR})$, $\mathbf{V}(1), \ldots, \mathbf{V}(2^{nR})$, each $\mathcal{N}(\mathbf{I}_n)$, independently. Reveal these sequences to all parties.
- Sending a sequence of messages $m_1, \ldots, m_K$, (each m_k in $\{1, \ldots, 2^{nR}\}$), requires $K+1$ blocks of length n. Fix $m_0 = m_{K+1} = 1$.
- During block k, ($k \in \{1, \ldots, K+1\}$), the encoder transmits $\sqrt{\bar{\alpha}P}\mathbf{U}(m_k) + \sqrt{\alpha P}\mathbf{V}(m_{k-1})$; the relay transmits $\sqrt{P_r}\mathbf{V}(\hat{m}_{k-1})$, where $\hat{m}_{k-1}$ is the relay's estimate of m_{k-1} from

$$\mathbf{Y}_r(k-1) = \sqrt{\bar{\alpha}P}\mathbf{U}(m_{k-1}) + \sqrt{\alpha P}\mathbf{V}(m_{k-2}) + \mathbf{Z}_r(k-1).$$

(green: already known at the relay)

- If $R < C(\bar{\alpha}P; N_r)$, then $\hat{m}_{k-1} = m_{k-1}$ with high probability; assume this is indeed the case.

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Decoding — Single Receiver

- The receiver, at the end of block k, ($k = 2, \ldots, K+1$) decodes m_{k-1} from the last two received blocks:

$$\mathbf{Y}(k-1) = \sqrt{\bar{\alpha}P}\mathbf{U}(m_{k-1}) + \sqrt{\alpha P}\mathbf{V}(m_{k-2}) + \sqrt{P_r}\mathbf{V}(m_{k-2}) + \mathbf{Z}(k-1)$$
$$\mathbf{Y}(k) = \sqrt{\bar{\alpha}P}\mathbf{U}(m_k) + \sqrt{\alpha P}\mathbf{V}(m_{k-1}) + \sqrt{P_r}\mathbf{V}(m_{k-1}) + \mathbf{Z}(k).$$

(green: already known, blue: signal, red: noise).

- This is possible if

$$R < C\left(\begin{bmatrix}\bar{\alpha}P & 0 \\ 0 & (\sqrt{\alpha P} + \sqrt{P_r})^2\end{bmatrix}; \begin{bmatrix}N & 0 \\ 0 & \bar{\alpha}P + N\end{bmatrix}\right)$$
$$= C(P + P_r + 2\sqrt{\alpha P P_r}; N).$$

Encoding — Two receivers

- Generate $\mathbf{U}^s(1), \ldots, \mathbf{U}^s(2^{nR^s})$, $\mathbf{V}^s(1), \ldots, \mathbf{V}^s(2^{nR^s})$, $\mathbf{U}^w(1), \ldots, \mathbf{U}^w(2^{nR^w})$, $\mathbf{V}^w(1), \ldots, \mathbf{V}^w(2^{nR^w})$, each $\mathcal{N}(\mathrm{I}_n)$, independently. Reveal these sequences to all parties.
- Sending a sequence of message pairs

$$(m_1^s, m_1^w), \ldots, (m_K^s, m_K^w) \quad (\text{each } m_k^\lambda \text{ in } \{1, \ldots, 2^{nR^\lambda}\})$$

requires $K + 2$ blocks. Fix $m_0^\lambda = m_{K+1}^\lambda = m_{K+2}^\lambda = 1$.

- During block k, ($k \in \{1, \ldots, K + 2\}$), the encoder transmits

$$\sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m_k^s) + \sqrt{\alpha\theta P}\mathbf{V}^s(m_{k-1}^s) + \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m_k^w) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m_{k-1}^w)$$

Encoding — Two Receivers

- The relay transmits

$$\sqrt{\theta_r P_r}\mathbf{V}^s(\hat{m}^s_{k-1}) + \sqrt{\bar{\theta}_r P_r}\mathbf{V}^w(\hat{m}^w_{k-1}),$$

  where $(\hat{m}^s_{k-1}, \hat{m}^w_{k-1})$ is the relay's estimate of (m^s_{k-1}, m^w_{k-1}) from

$$\begin{aligned}\mathbf{Y}_r(k-1) = &\sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m^s_{k-1}) + \sqrt{\alpha\theta P}\mathbf{V}^s(m^s_{k-2}) \\ &+ \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m^w_{k-1}) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m^w_{k-2}) + \mathbf{Z}_r(k-1).\end{aligned}$$

- If $(R^s, R^w) \in C_{\text{MAC}}(\bar{\alpha}\theta P, \bar{\beta}\bar{\theta}P; N_r)$, then $\hat{m}^\lambda_{k-1} = m^\lambda_{k-1}$ with high probability; assume this is indeed the case.

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Decoding — Weak Receiver

- At the end of block k, $(k = 2, \ldots, K+1)$ the weak receiver decodes m^w_{k-1} from the last two received blocks:

$$
\begin{aligned}
\mathbf{Y}_w(k-1) = {} & \sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m^s_{k-1}) + \sqrt{\alpha\theta P}\mathbf{V}^s(m^s_{k-2}) \\
& + \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m^w_{k-1}) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m^w_{k-2}) \\
& + \sqrt{\theta_r P_r}\mathbf{V}^s(m^s_{k-2}) + \sqrt{\bar{\theta}_r P_r}\mathbf{V}^w(m^w_{k-2}) + \mathbf{Z}_w(k-1) \\
\mathbf{Y}_w(k) = {} & \sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m^s_{k}) + \sqrt{\alpha\theta P}\mathbf{V}^s(m^s_{k-1}) \\
& + \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m^w_{k}) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m^w_{k-1}) \\
& + \sqrt{\theta_r P_r}\mathbf{V}^s(m^s_{k-1}) + \sqrt{\bar{\theta}_r P_r}\mathbf{V}^w(m^w_{k-1}) + \mathbf{Z}_w(k).
\end{aligned}
$$

Decoding — Weak Receiver

This is possible if

$$R^w < C\left(\begin{bmatrix}\bar\beta\bar\theta P & 0 \\ 0 & (\sqrt{\beta\bar\theta P}+\sqrt{\bar\theta_r P_r})^2\end{bmatrix}; \begin{bmatrix}N_w & 0 \\ 0 & N_w\end{bmatrix}\right.$$
$$\left. + \begin{bmatrix}\bar\alpha\theta P + (\sqrt{\alpha\theta P}+\sqrt{\theta_r P_r})^2 & 0 \\ 0 & \bar\alpha\theta P + (\sqrt{\alpha\theta P}+\sqrt{\theta_r P_r})^2 + \bar\beta\bar\theta P\end{bmatrix}\right)$$
$$= C(\bar\theta P + \bar\theta_r P_r + 2\sqrt{\beta\bar\theta\bar\theta_r P P_r};$$
$$\theta P + \theta_r P_r + 2\sqrt{\alpha\theta\theta_r P P_r} + N_w).$$

Decoding — Strong Receiver

- At the end of block k, ($k = 3, \dots, K+2$) the strong receiver decodes m^w_{k-1} from the last two received blocks $\mathbf{Y}_s(k-1), \mathbf{Y}_s(k)$, and then decodes m^s_{k-2} from:

$$
\begin{aligned}
\mathbf{Y}_s(k-2) = {} & \sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m^s_{k-2}) + \sqrt{\alpha\theta P}\mathbf{V}^s(m^s_{k-3}) \\
& + \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m^w_{k-2}) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m^w_{k-3}) \\
& + \sqrt{\theta_r P_r}\mathbf{V}^s(m^s_{k-3}) + \sqrt{\bar{\theta}_r P_r}\mathbf{V}^w(m^w_{k-3}) + \mathbf{Z}_s(k-2) \\
\mathbf{Y}_s(k-1) = {} & \sqrt{\bar{\alpha}\theta P}\mathbf{U}^s(m^s_{k-1}) + \sqrt{\alpha\theta P}\mathbf{V}^s(m^s_{k-2}) \\
& + \sqrt{\bar{\beta}\bar{\theta}P}\mathbf{U}^w(m^w_{k-1}) + \sqrt{\beta\bar{\theta}P}\mathbf{V}^w(m^w_{k-2}) \\
& + \sqrt{\theta_r P_r}\mathbf{V}^s(m^s_{k-2}) + \sqrt{\bar{\theta}_r P_r}\mathbf{V}^w(m^w_{k-2}) + \mathbf{Z}_s(k-1).
\end{aligned}
$$

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Decoding — Strong Receiver

This is possible if

$$R^s < C\left(\begin{bmatrix}\bar{\alpha}\theta P & 0 \\ 0 & (\sqrt{\alpha\theta P}+\sqrt{\theta_r P_r})^2\end{bmatrix}; \begin{bmatrix}N_s & 0 \\ 0 & \bar{\alpha}\theta P + N_s\end{bmatrix}\right)$$
$$= C(\theta P + \theta_r P_r + 2\sqrt{\alpha\theta\theta_r P P_r}; N_s).$$

An achievable region

Theorem

All (R^s, R^w) satisfying

$$
\begin{aligned}
R^s &\leq C(\theta P + \theta_r P_r + 2\sqrt{\alpha\theta\theta_r P P_r}; N_s) \\
R^w &\leq C(\bar{\theta} P + \bar{\theta}_r P_r + 2\sqrt{\beta\bar{\theta}\bar{\theta}_r P P_r}; \\
&\qquad \theta P + \theta_r P_r + 2\sqrt{\alpha\theta\theta_r P P_r} + N_w) \\
(R^s, R^w) &\in C_{MAC}(\bar{\alpha}\theta P, \bar{\beta}\bar{\theta} P; N_r)
\end{aligned}
$$

for some $(\alpha, \beta, \theta, \theta_r) \in [0,1]^4$ are achievable.

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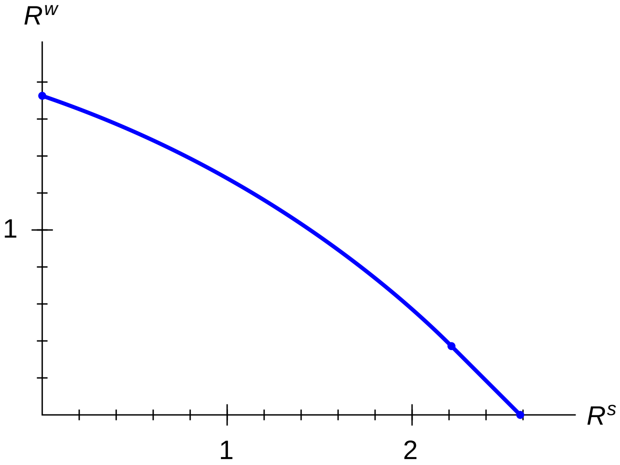
Numerical Example

Inner bound for $P = 5$, $P_r = 10$, $N_r = 1$, $N_s = 2.5$, $N_w = 11$



Outer bound for Physically 'Upgraded' Relay

Suppose that the receivers are physically degraded with respect to the relay:

$$\begin{aligned} Y_r &= X + Z_r \\ Y_s &= X + X_r + Z_r + \tilde{Z}_s \qquad \tilde{Z}_s \perp\!\!\!\perp Z_r \\ Y_w &= X + X_r + Z_r + \tilde{Z}_w \qquad \tilde{Z}_w \perp\!\!\!\perp Z_r \end{aligned}$$

We can assume (without loss of generality)

$$Y_w = Y_s + \tilde{Z} \qquad \tilde{Z} \perp\!\!\!\perp Y_s$$

Outer bound for Physically ‘Upgraded’ Relay

Given a system with blocklength n that operates at powers P and P_r, rate (R_s, R_w) and with low probability of error, let

$$\rho = \frac{E[\langle \mathbf{X}(W_s, W_w), \mathbf{X}_r \rangle]}{n\sqrt{PP_r}}$$

so that $\frac{1}{n}E[\|\mathbf{X} + \mathbf{X}_r\|^2] = P + P_r + 2\rho\sqrt{PP_r}$. Then

$$\begin{aligned} nR_w &= H(W_w) \\ &\doteq I(W_w; \mathbf{Y}_w) \\ &= h(\mathbf{Y}_w) - h(\mathbf{Y}_w | W_w) \\ &\leq \frac{n}{2}\log[2\pi e(N_w + P + P_r + 2\rho\sqrt{PP_r})] - h(\mathbf{Y}_w | W_w). \end{aligned}$$

Outer bound for Physically 'Upgraded' Relay

Since $h(Z_w) \leq h(\mathbf{Y}_w|W_w) \leq h(\mathbf{Y}_w)$ we can define, $\delta \in [0,1]$ by

$$\frac{n}{2}\log[2\pi e(N_w + \delta(P + P_r + 2\rho\sqrt{PP_r}))] = h(\mathbf{Y}_w|W_w)$$

so that

$$R_w \leq C(\bar{\delta}(P + P_r + 2\rho\sqrt{PP_r}); N_w + \delta(P + P_r + 2\rho\sqrt{PP_r})).$$

Moreover,

$$\begin{aligned} nR_s &= H(W_s|W_w) \doteq I(W_s; \mathbf{Y}_s|W_w) \\ &= h(\mathbf{Y}_s|W_w) - h(Z_s) \end{aligned}$$

The entropy-power inequality yields

$$h(\mathbf{Y}_s|W_w) \leq \frac{n}{2}\log[2\pi e(N_s + \delta(P + P_r + 2\rho\sqrt{PP_r}))]$$

and $R_s \leq C(\delta(P + P_r + 2\rho\sqrt{PP_r}); N_s)$.

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Outer bound for Physically 'Upgraded' Relay

Also,

$$
\begin{aligned}
R_s + R_w &\leq n^{-1} \sum_i I(X_i; Y_{r,i}, Y_{s,i}, Y_{w,i} | X_{r,i}) \\
&= n^{-1} \sum_i I(X_i; Y_{r,i} | X_{r,i}) && \text{physical degradation} \\
&= n^{-1} \sum_i h(Y_{r,i} | X_{r,i}) - h(Z_{r,i}) \\
&\leq n^{-1} \sum_i h(Y_{r,i} - \rho \sqrt{P/P_r} X_{r,i}) - h(Z_{r,i}) \\
&\leq n^{-1} \sum_i C\left(E[X_i - \rho \sqrt{P/P_r} X_{r,i}]^2; N_r\right) \\
&\leq C\left(n^{-1} \sum_i E[X_i - \rho \sqrt{P/P_r} X_{r,i}]^2; N_r\right) \\
&= C((1-\rho^2)P; N_r).
\end{aligned}
$$

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Outer bound for Physically ‘Upgraded’ Relay

Replacing ρ with $|\rho|$ increases all right hand sides, thus:

Theorem

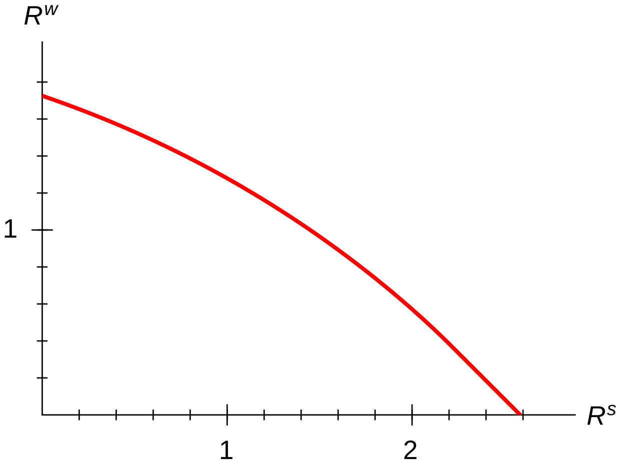
When the receivers are physically degraded with respect to the relay, every achievable (R_s, R_w) satisfies

$$\begin{aligned} R_s &\leq C(\delta[P + P_r + 2\sqrt{\gamma P P_r}]; N_s) \\ R_w &\leq C(\bar{\delta}[P + P_r + 2\sqrt{\gamma P P_r}]; \\ &\qquad \delta[P + P_r + 2\sqrt{\gamma P P_r}] + N_w) \\ R_s + R_w &\leq C(\bar{\gamma} P; N_r) \end{aligned}$$

for some $(\gamma, \delta) \in [0, 1]^2$.

Numerical Example

Outer bound for $P = 5$, $P_r = 10$, $N_r = 1$, $N_s = 2.5$, $N_w = 11$



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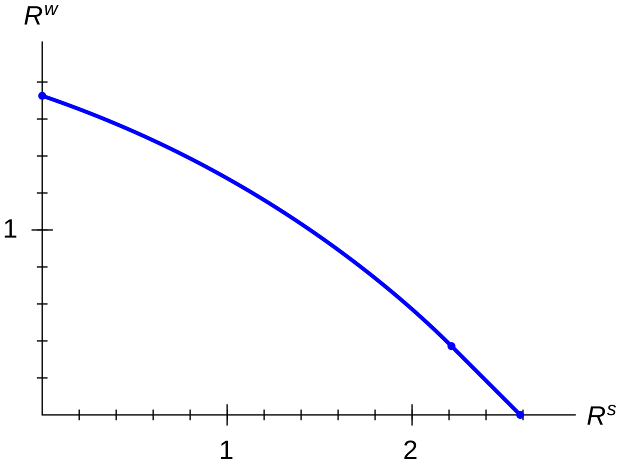

Capacity region

Theorem

The inner and outer bounds to the achievable rates given previously coincide.

Proof.

Ugly. □

Ugly proof steps:

- Suffices to show that the boundary of the outer bound is achievable.
- This boundary corresponds to γ, δ pairs for which the $R_s + R_w$ bound 'just' bites.
- For such γ, δ, one can explicitly find $\alpha, \beta, \theta, \theta_r$ in $[0, 1]^4$.

$$\theta_r = \delta \frac{P + P_r + 2\sqrt{\gamma P P_r}}{\gamma P + P_r + 2\sqrt{\gamma P P_r}} \frac{N_s - N_r}{N_s} \qquad \alpha = \gamma \frac{\theta_1}{\theta}$$

$$\theta = \frac{\delta[P + P_r + 2\sqrt{\gamma P P_r}] - \theta_r[P_r + 2\sqrt{\gamma P P_r}]}{P} \qquad \beta = \gamma \frac{\bar{\theta}_r}{\bar{\theta}}$$

Conclusion

The model is clearly unrealistic. But the solution is lucky.