Network Coding for networks of noisy & interfering channels

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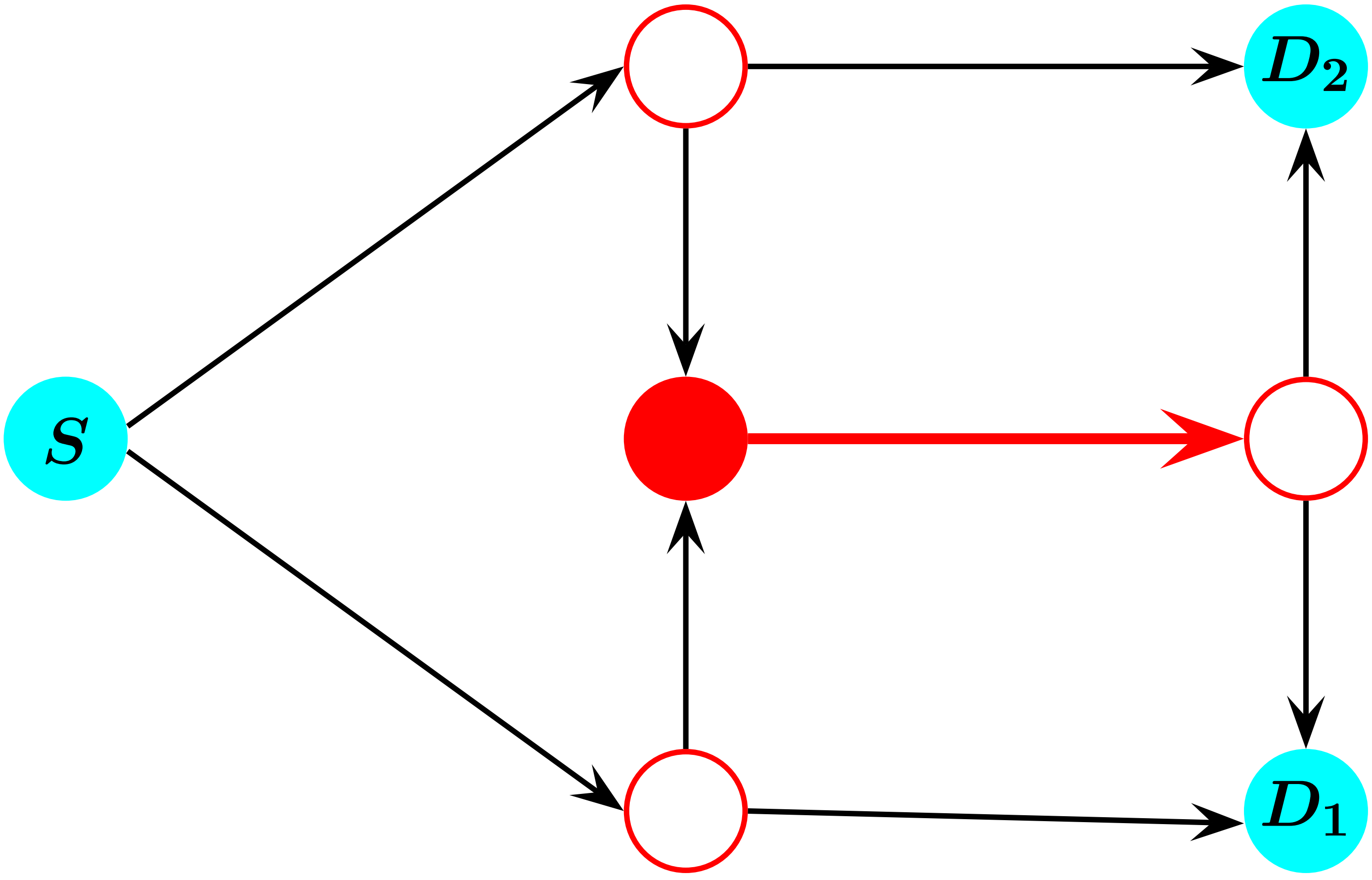
Motivation

- Communication network today are organized in large scale networks (Internet) where packets traverse multiple hops in order to reach the destination
- In Ad-hoc wireless networks, the number of hops scales as $\sqrt{\#\text{nodes}}$
- Each hop introduces errors, that become more pronounced as the number of hops increases
- Does processing at the intermediate node processing improve performance? If so, what kind of processing?

Motivation - cont.

Network Coding improves the end-to-end performance, even on network of noise-free links!



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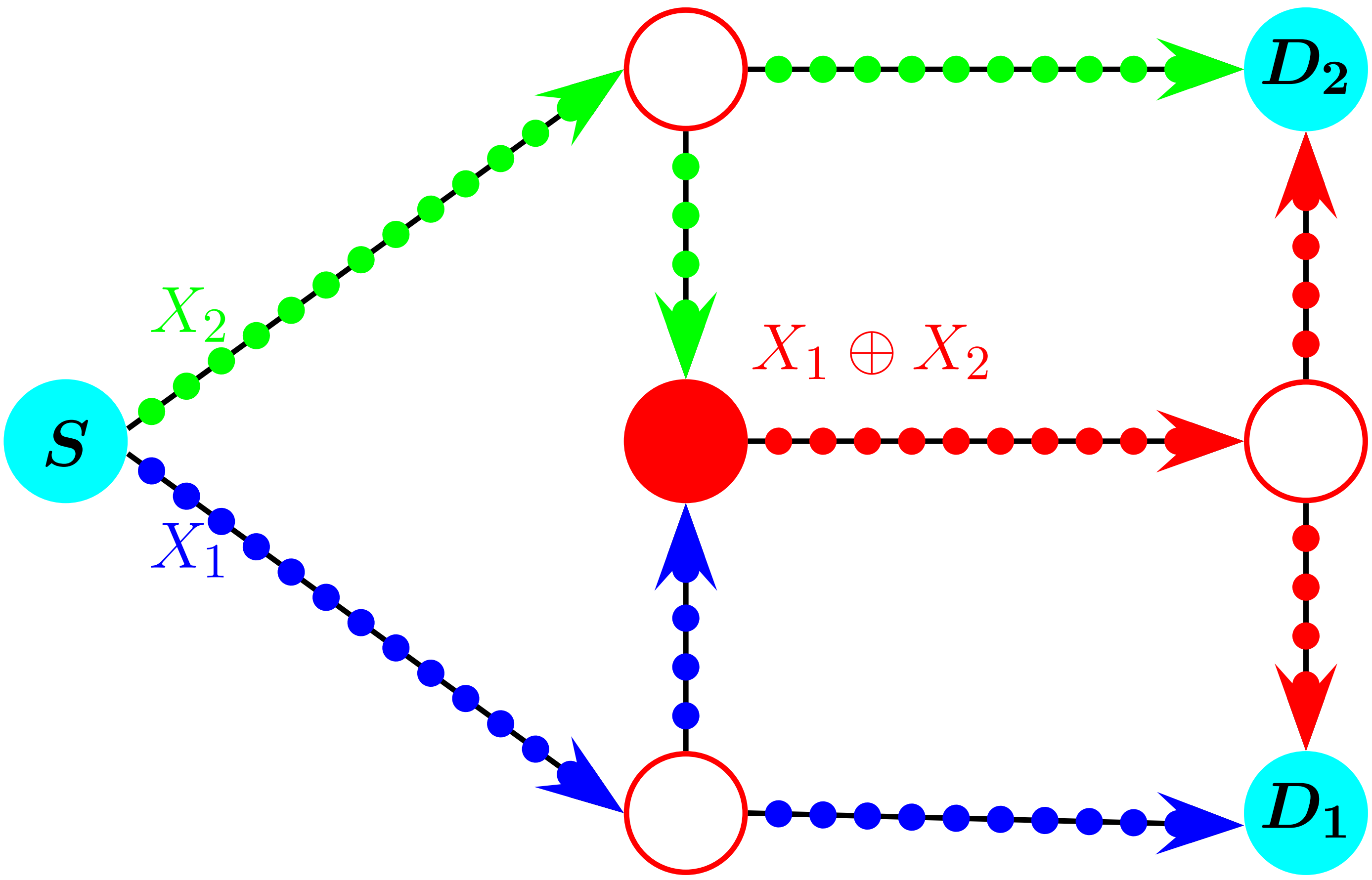

Motivation - cont.

What happens if

- the links are noisy?
- the channels interfere?
- the graph has cycles?

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  We'll comment on this at the end of this talk ...
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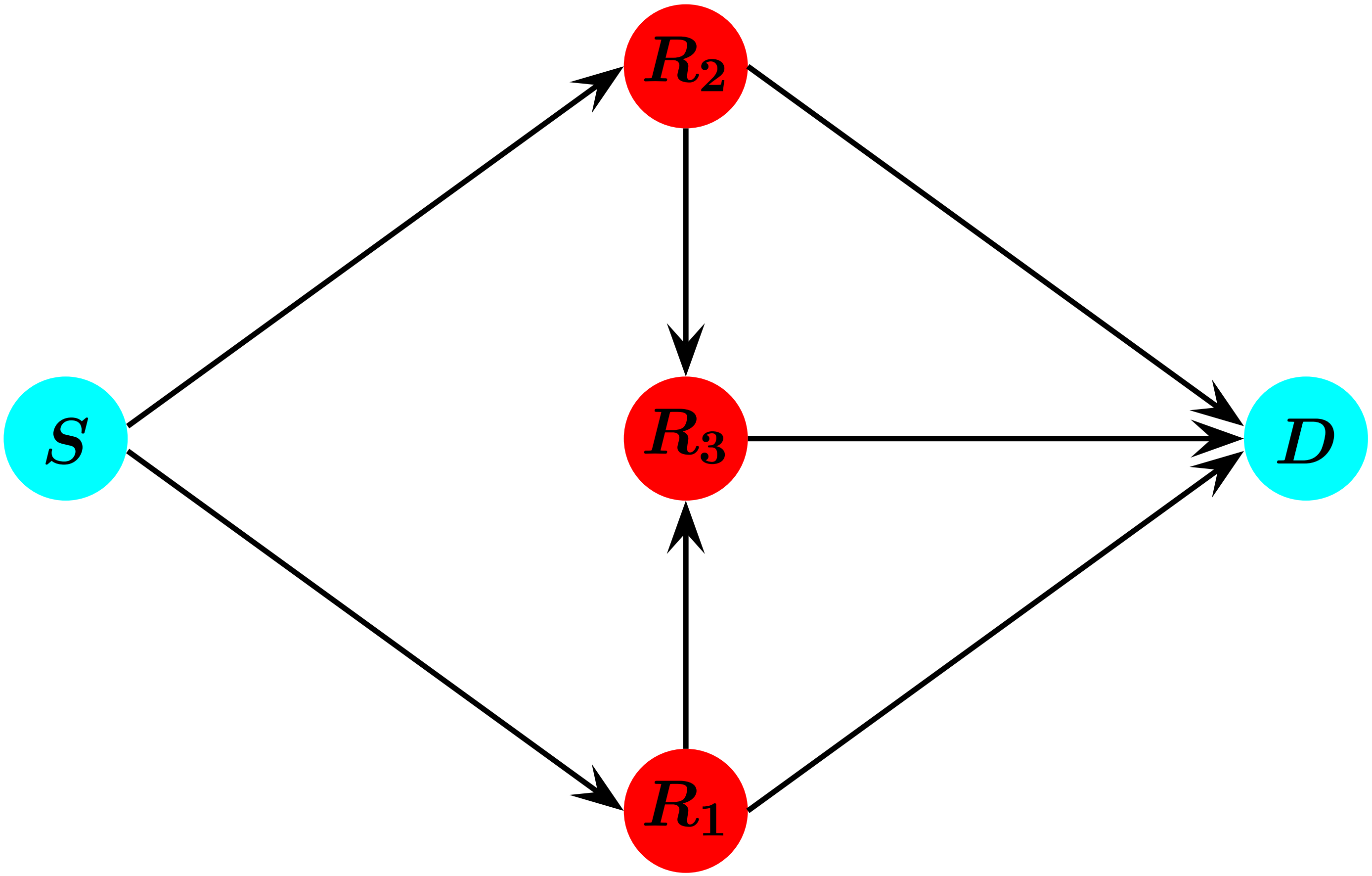
- the graph has cycles?
  Feedback: open problem ...

Example: Unicast (F&T ISIT'05)



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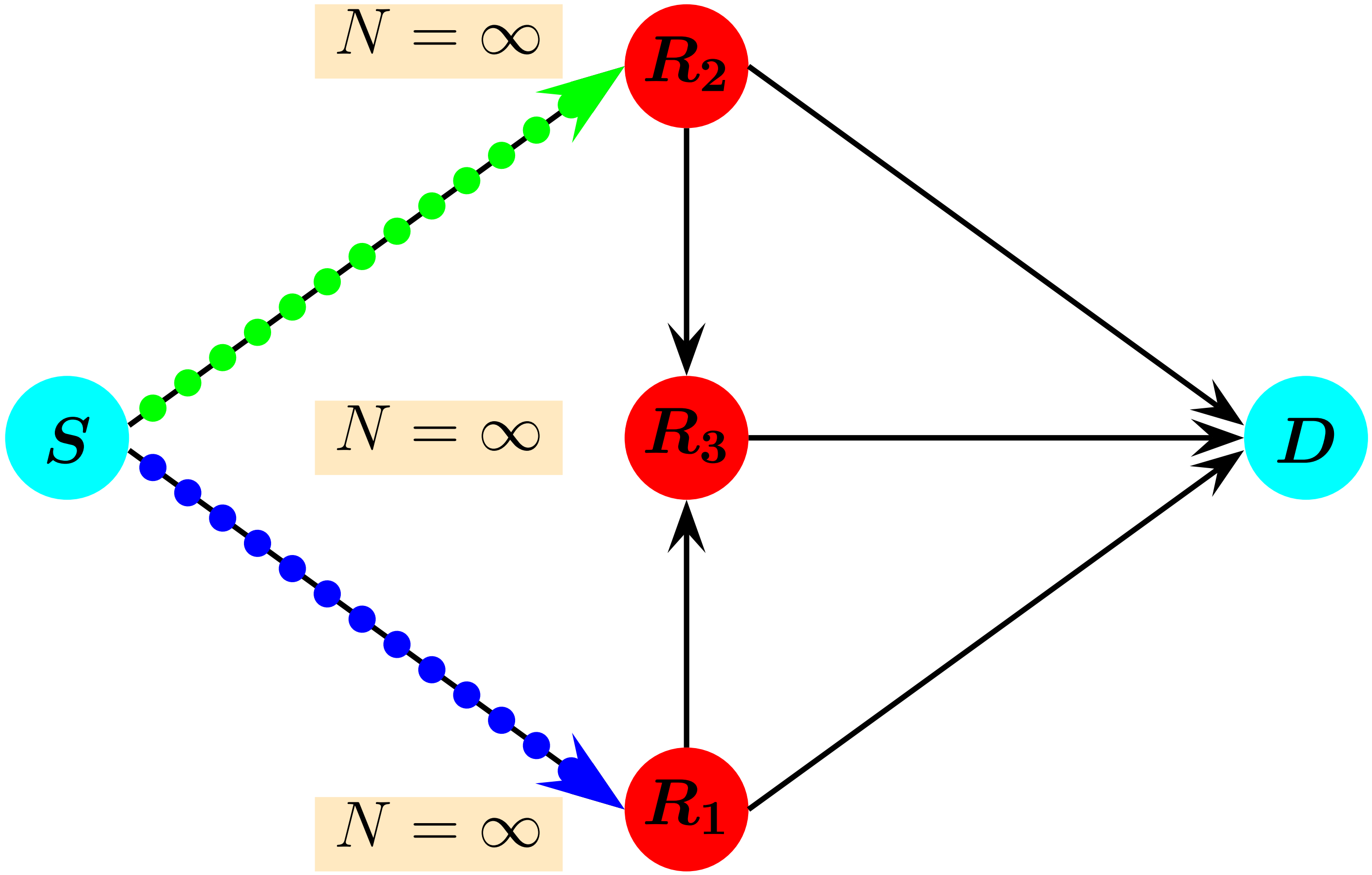

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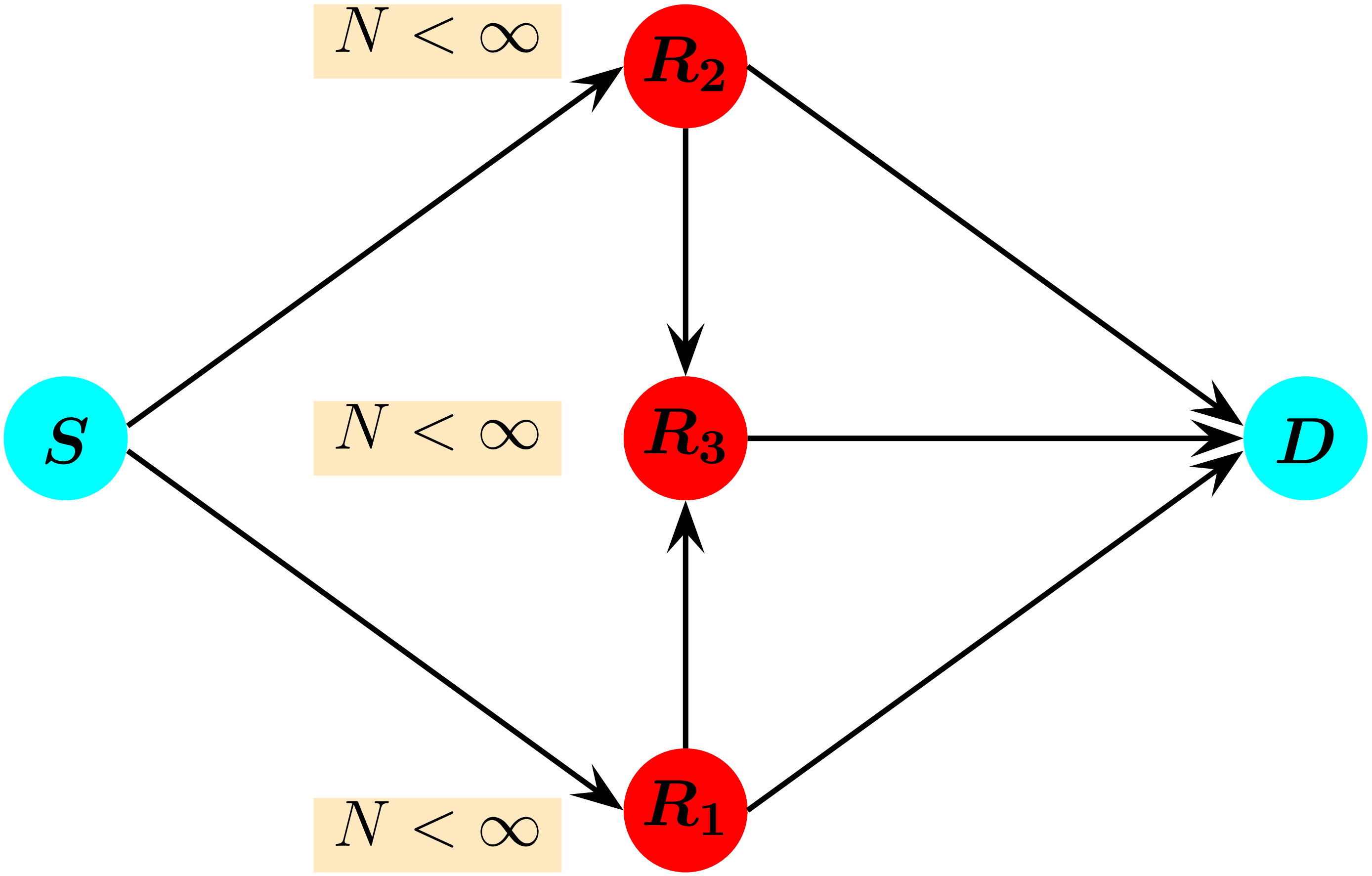

Network Model

- Hypothesis
  - Network of DMCs
  - Relay nodes can process blocks of finite length N only (well suited for packet oriented networks)
  - Source and Destination can perform coding and decoding of arbitrary complexity/length

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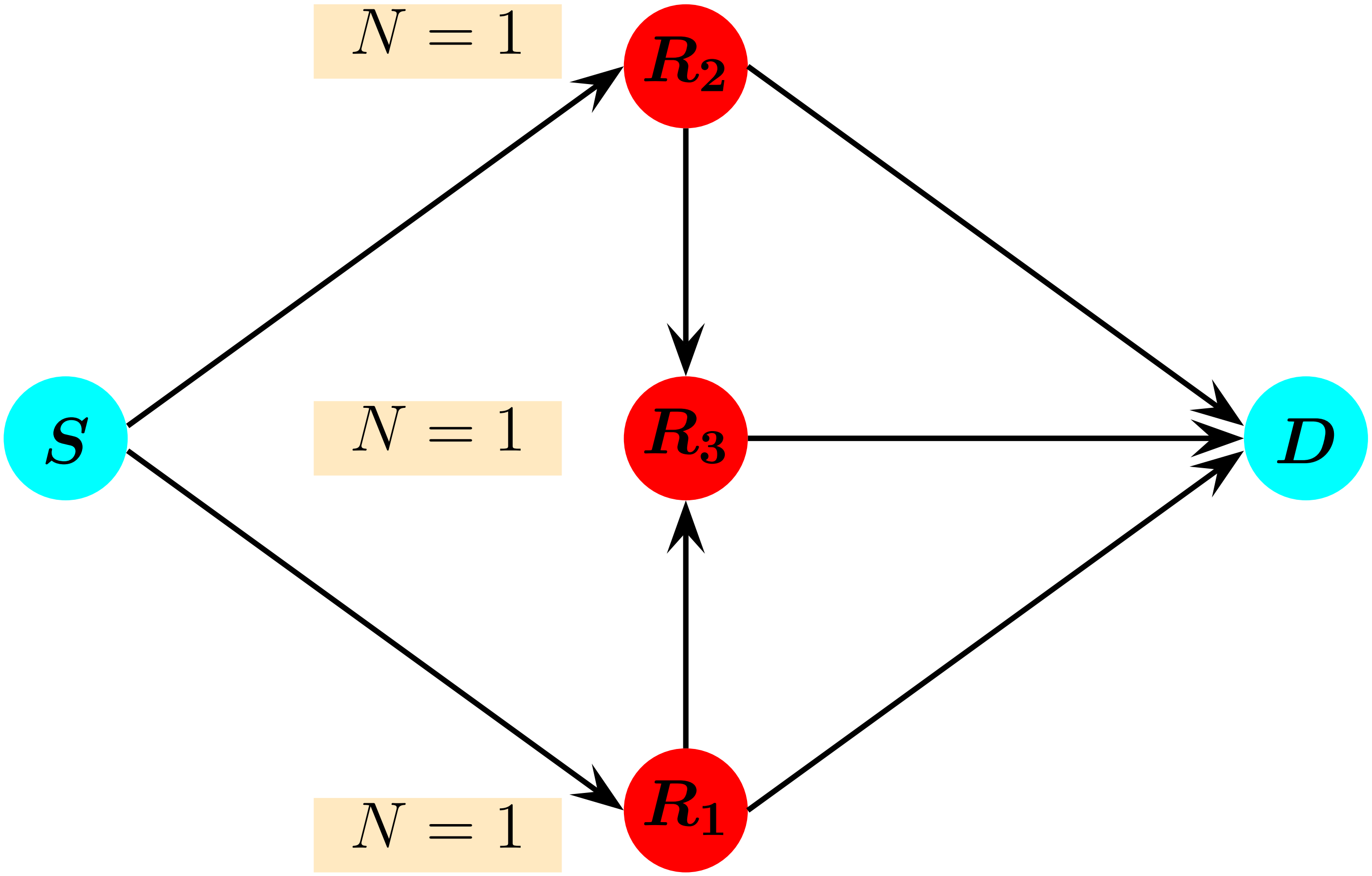
- Hypothesis
  - Network of DMCs
  - Relay nodes can process blocks of finite length N only (well suited for packet oriented networks)
  - Source and Destination can perform coding and decoding of arbitrary complexity/length
- Goal: Determine the capacity of the network
  - Does finite complexity processing improve over forwarding?
  - Properties of optimal intermediate processing
  - Scalability in large networks
  - Does N need to scale with the network size in order to achieve the "min-cut" capacity?

Example: Network of BSC

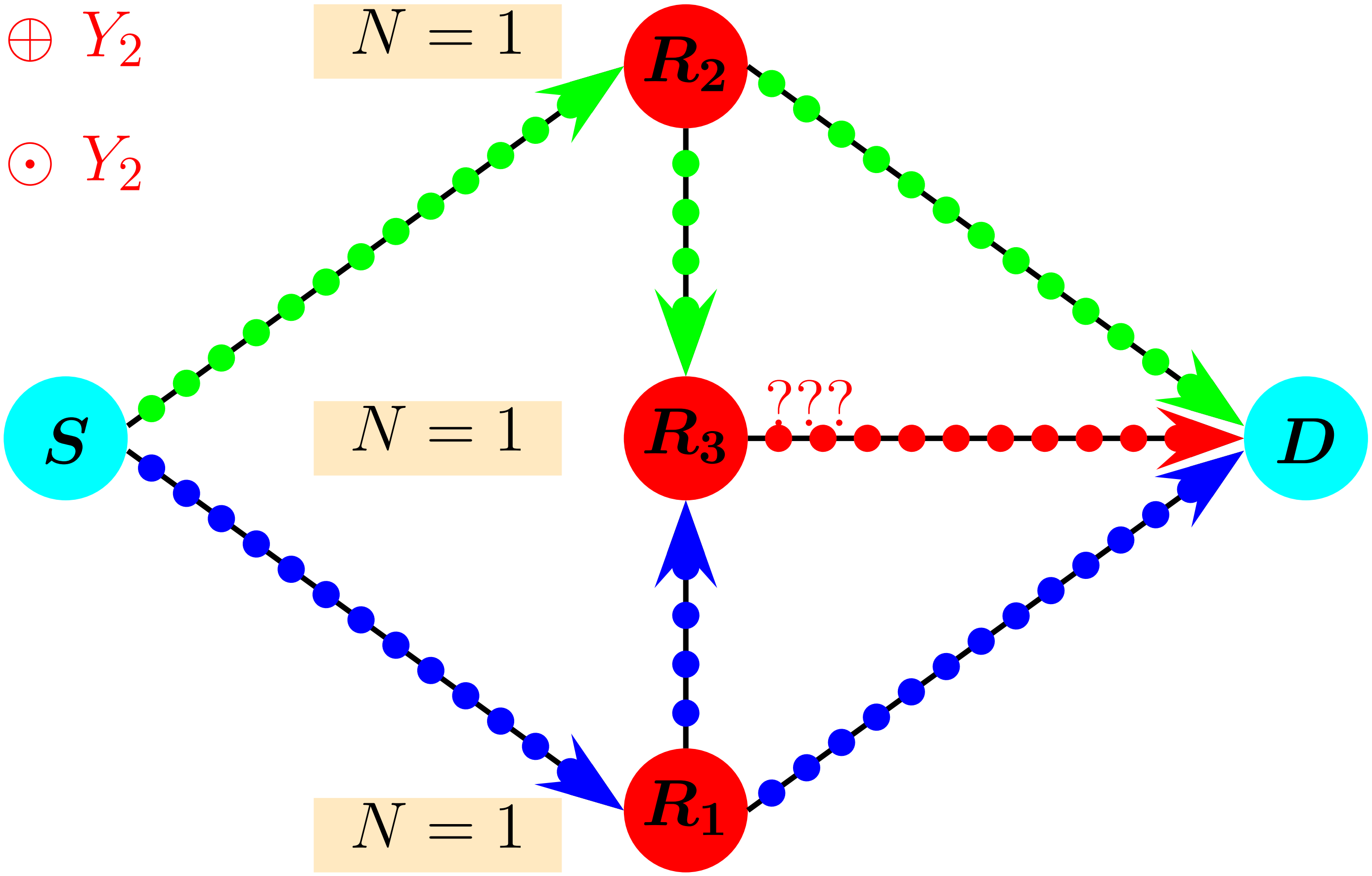


Example: Network of BSC

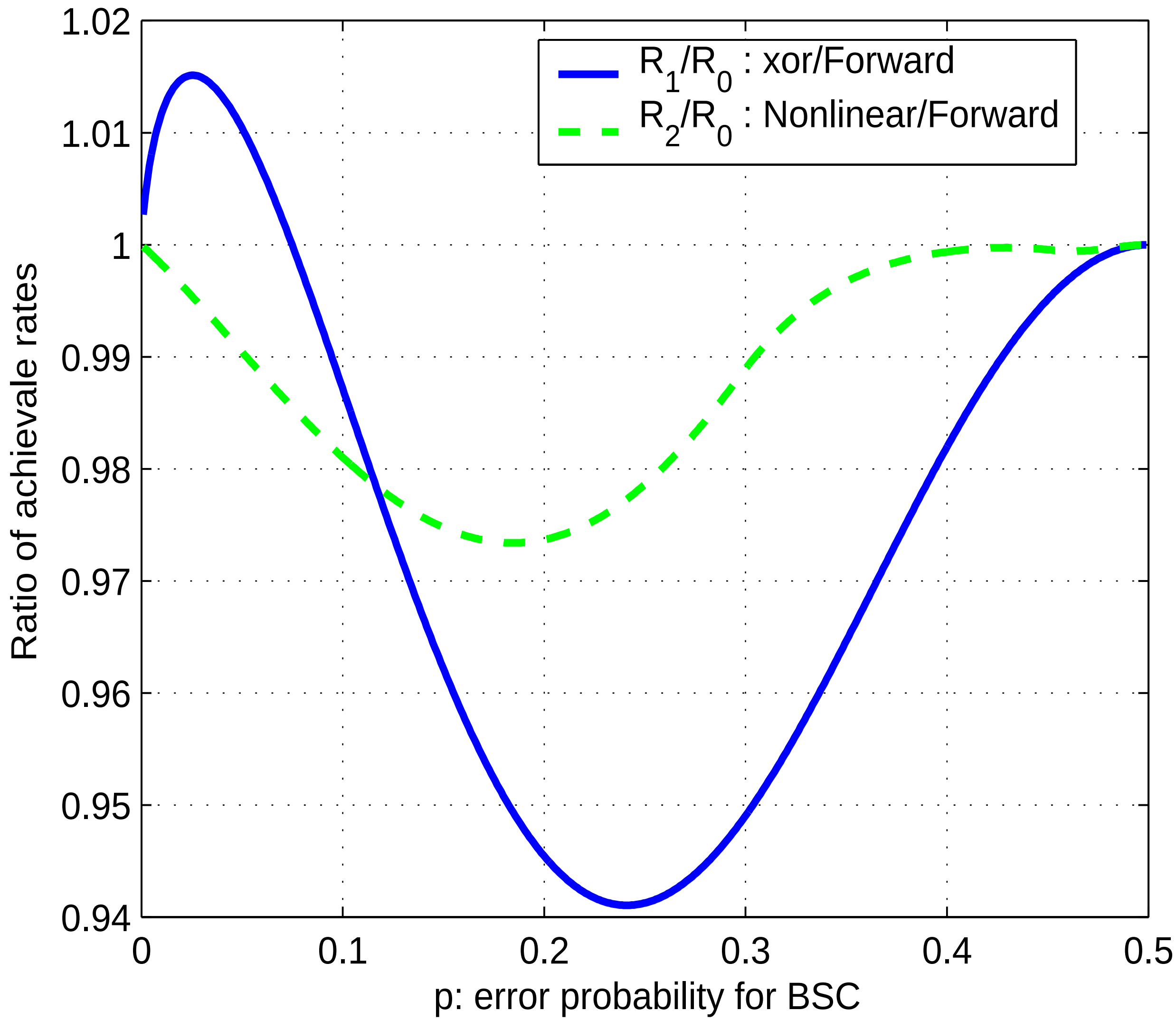
$??? = Y_1$ (or Y_2)

$??? = Y_1 \oplus Y_2$

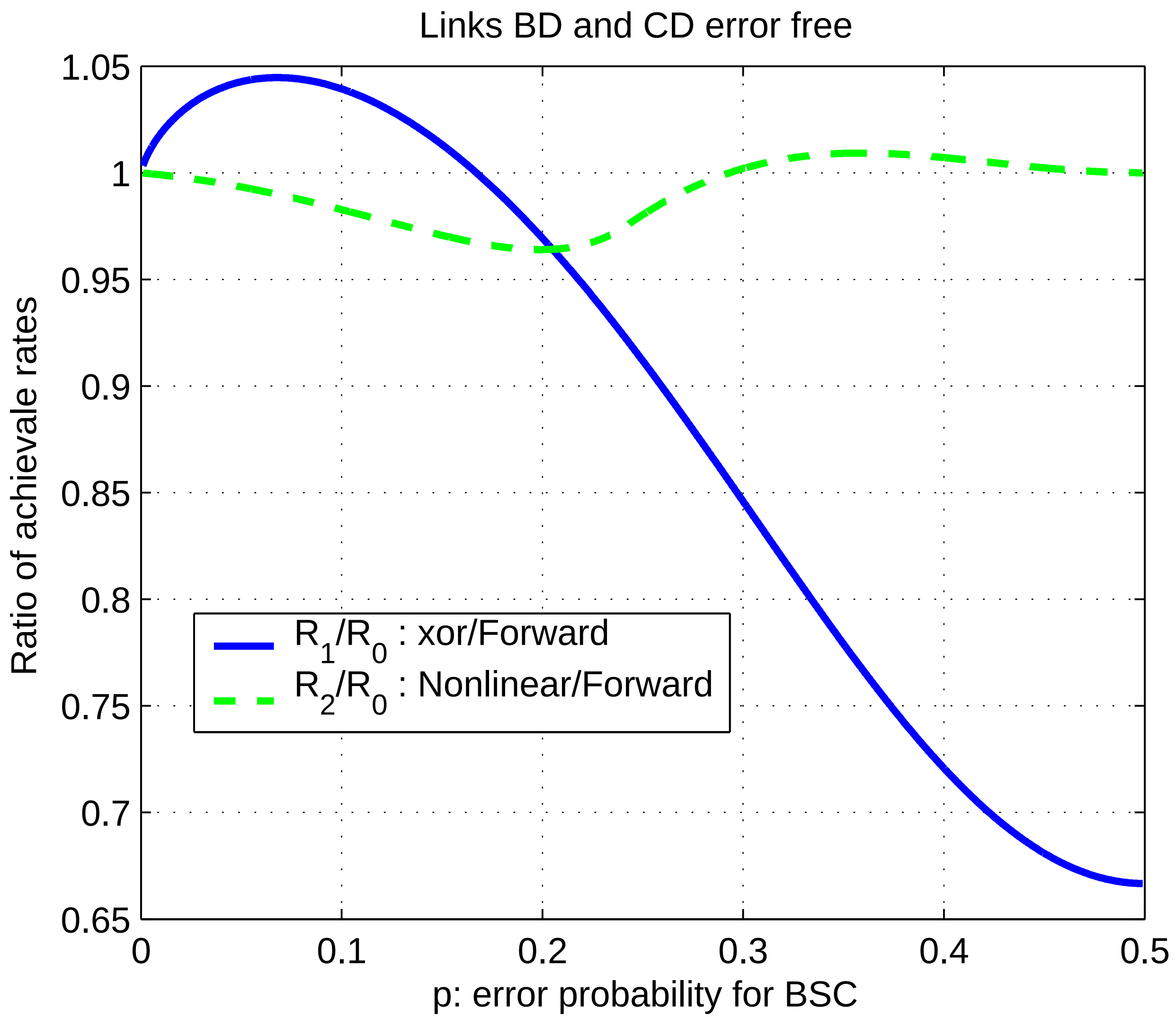
$??? = Y_1 \odot Y_2$



Example: all links are BSC(p)



Example: noiseless links among relays



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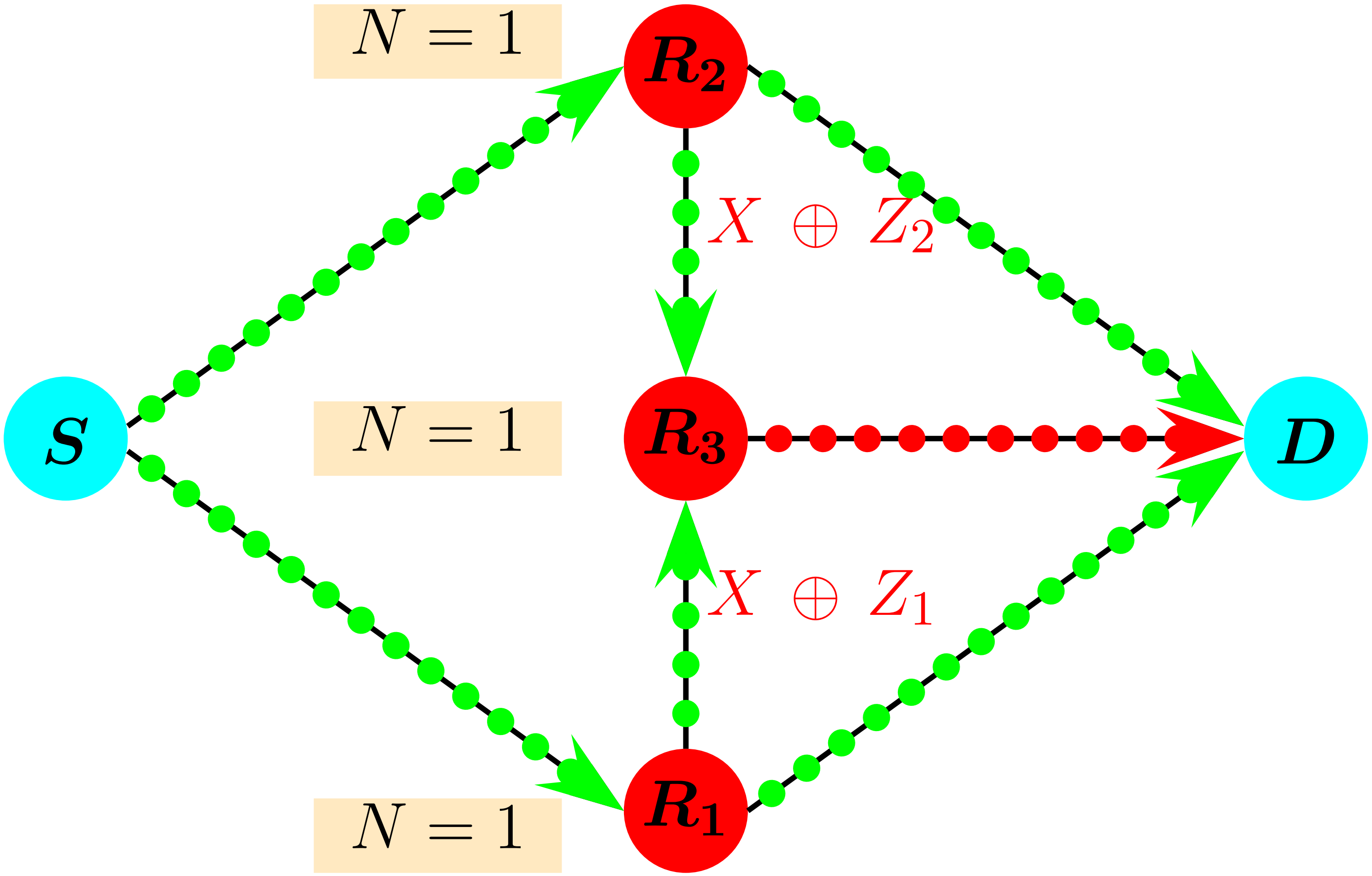

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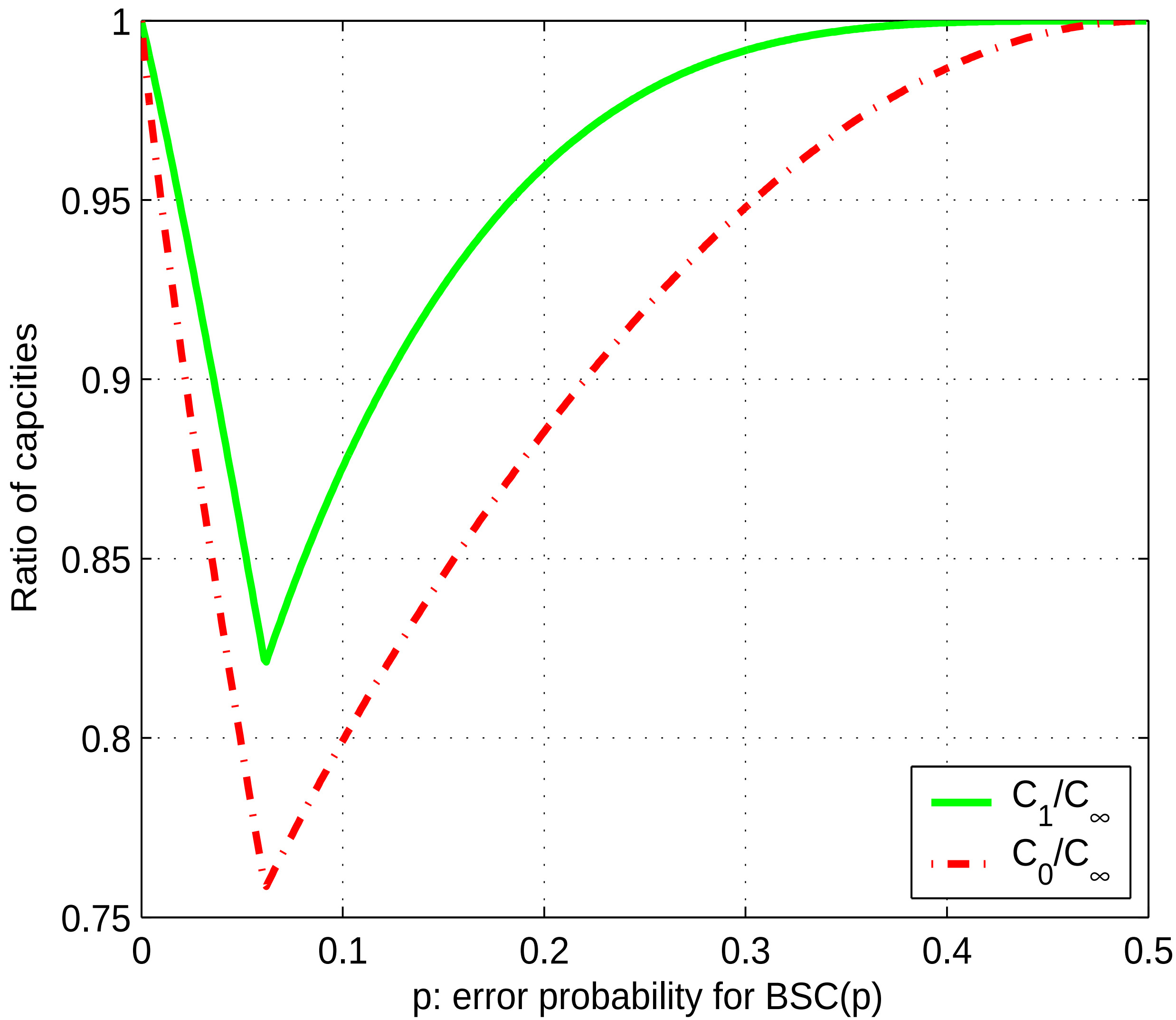
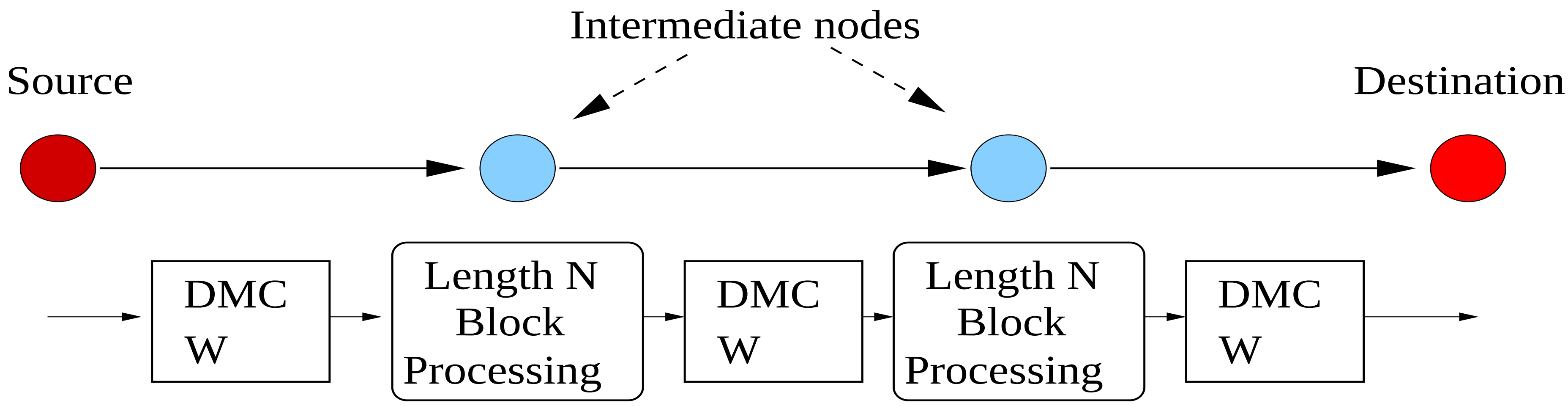


Line Networks



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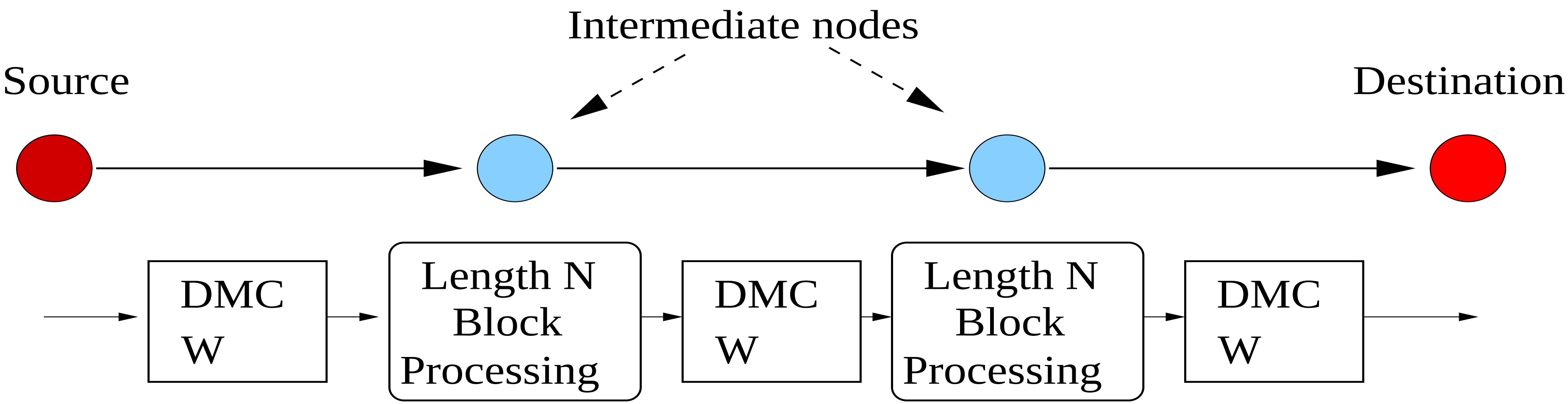

$$\boldsymbol{W}_{\mathrm{eq}} \triangleq \boldsymbol{W}^{\otimes N} \prod_{\ell=1}^{L-1} \left(\boldsymbol{M}_{\ell} \boldsymbol{W}^{\otimes N} \right) = \boldsymbol{W}_{\mathrm{eq}}(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{L-1})$$

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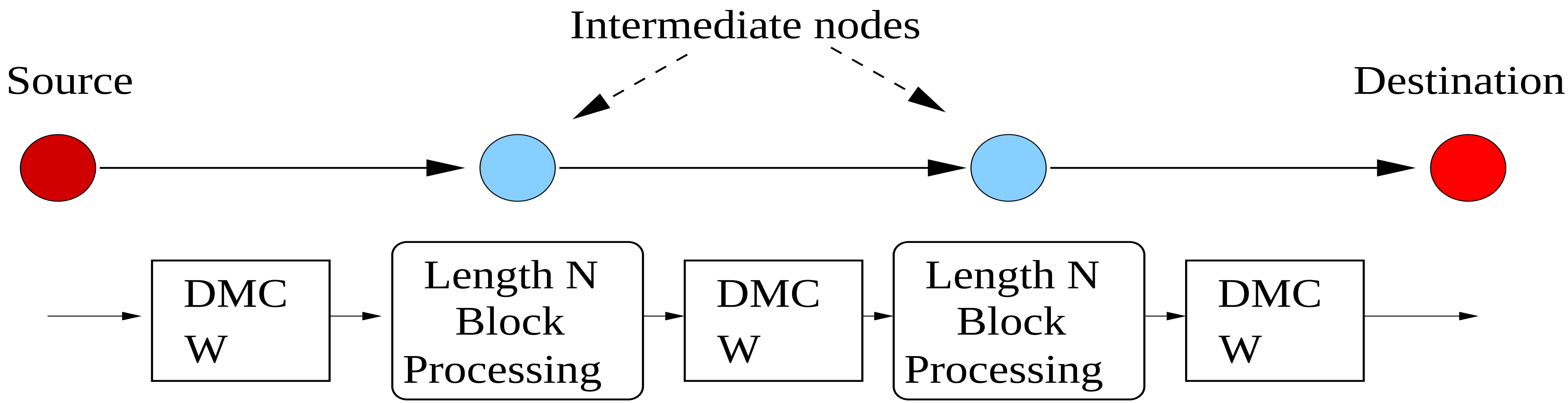

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$$C_{N,L}(\boldsymbol{W}) \triangleq \max_{\{\boldsymbol{M}_{\ell}\}_{\ell=1}^{L-1}} \max_{\boldsymbol{p}} \frac{1}{N} I\left(\boldsymbol{p}, \boldsymbol{W}_{\text{eq}}\right)$$

Line Networks: Main Results

- In general

$$\underbrace{\frac{1}{N}\log M_0(\boldsymbol{W}^{\otimes N})}_{\text{zero-error achievable rate}} \leq C_{N,L}(\boldsymbol{W}) \leq \underbrace{C(\boldsymbol{W})}_{\text{min-cut capacity}}$$

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- Finite N & $L \to \infty$ (Allerton 2005)

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Line Networks: Main Results - cont.

- For $N = \Theta(\log L)$ & $L \to \infty$

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    - $N = \Theta(\log L)$ *sufficient* for all $\alpha \in [0, 1]$ (ISITA 2004)

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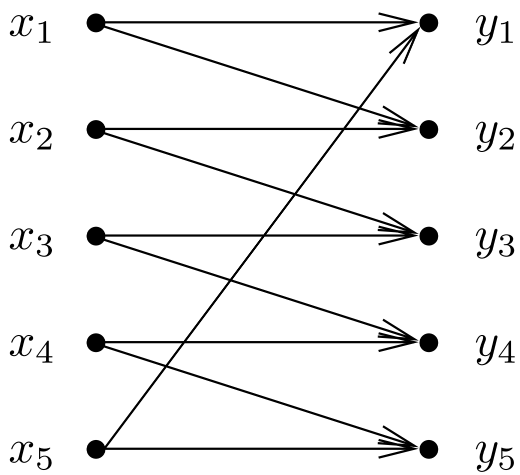
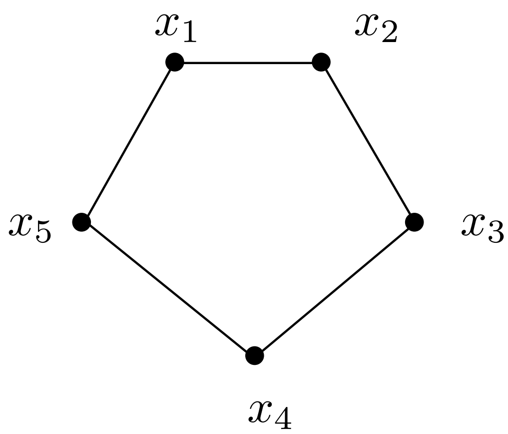
  - $N = \Theta(\log L)$ *sufficient* for all $\alpha \in [0, 1]$ (ISITA 2004)
  - $N = \Theta(\log L)$ *necessary* for all $\alpha \in [\beta, 1]$ (ISIT 2006)

$$\beta = \frac{\lim_{m\to\infty} \frac{1}{m} \log \operatorname{rank}(\boldsymbol{A}_m) - C_0(\boldsymbol{W})}{C(\boldsymbol{W}) - C_0(\boldsymbol{W})} \geq 0,$$

but we conjecture $\beta = 0$.

Example: The Pentagon Channel





$$M_0\big(\boldsymbol{W}\big) = 2, \quad M_0\big(\boldsymbol{W}^{\otimes 2}\big) = 5$$

$$C_0(\boldsymbol{W}) = \frac{1}{2}\log 5 \quad \text{achieved by} \quad N = 2$$

Example: The Pentagon Channel

- For an infinite cascade of "pentagon" channels

$$\lim_{L\to\infty} C_{1,L}(\boldsymbol{W}) = \log 2, \qquad \lim_{L\to\infty} C_{2,L}(\boldsymbol{W}) = \frac{1}{2}\log 5$$

  i.e., $N = 2$ is optimal if N is restricted to be finite.

- With forwarding

$$\lim_{L\to\infty} C(\boldsymbol{W}^L) = \log 1 = 0,$$

  and this limit is approached exponentially fast.

- Intermediate processing, as simple as one-symbol processing, is *necessary* if a non-vanishing throughput is to be achieved in a long line network.

Example: The Pentagon Channel

About

$$\boldsymbol{W}^{\otimes m} = \delta_m \, \boldsymbol{A}_m + (1-\delta_m) \, \boldsymbol{B}_m$$

we can find

$$\operatorname{rank}(\boldsymbol{A}_1) = 3$$

$$\operatorname{rank}(\boldsymbol{A}_2) = 8 < \operatorname{rank}(\boldsymbol{A}_1)^2 = 9$$

$$\beta \leq \frac{\frac{1}{2}\log 8 - \frac{1}{2}\log 5}{C(\boldsymbol{W}) - \frac{1}{2}\log 5}$$

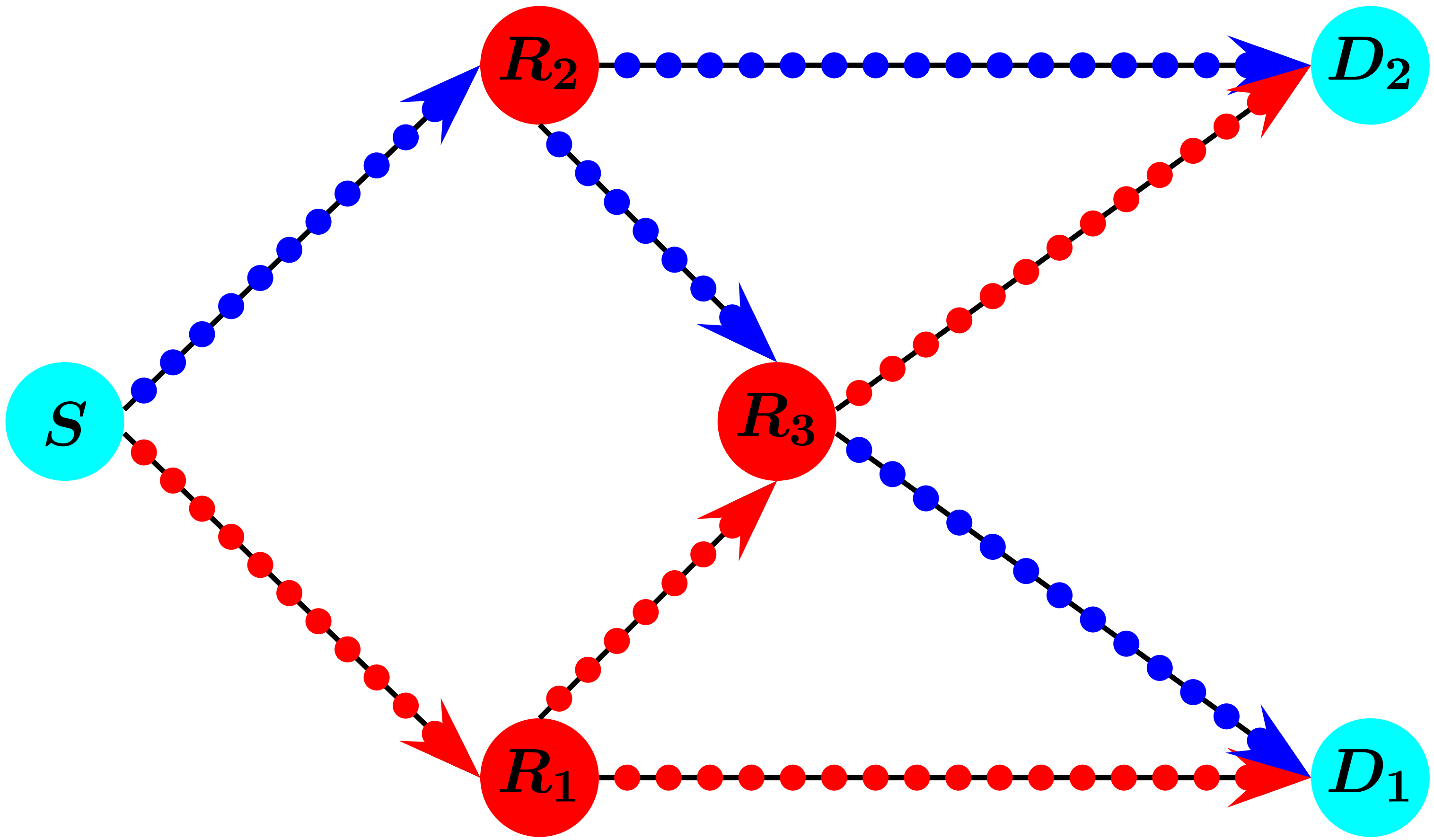
Is logarithmic growth is necessary for $0 < \alpha < \beta$?

General Networks?

... work in progress ...

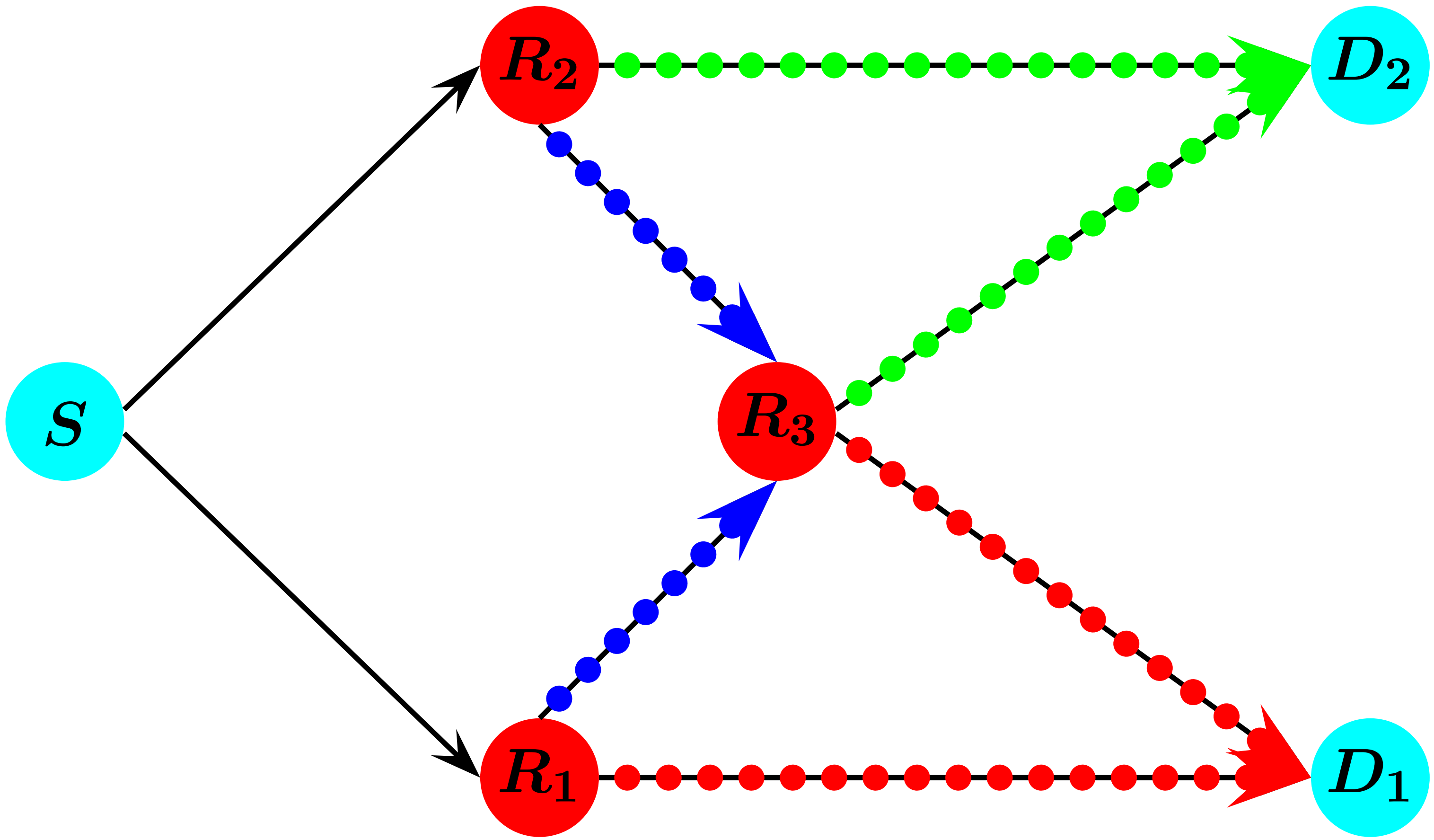
Networks of Interfering Links

Non-interfering links

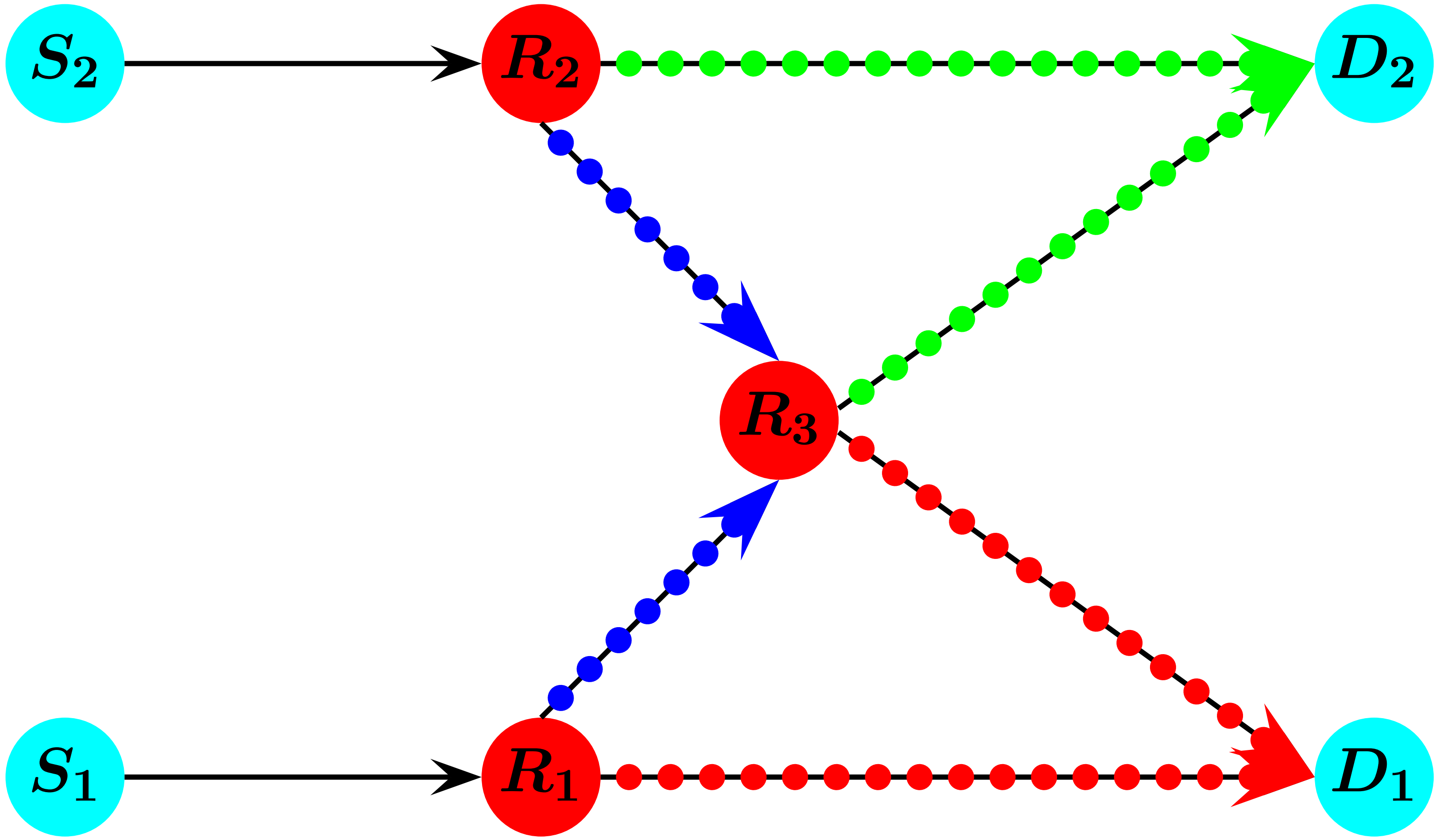


Networks of Interfering Links

Interfering links



The bow-tie example



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Conclusions

- The “classical” Network Coding model implicitly assumes channel orthogonalization at MAC, the use of capacity achieving codes at PHY.
  Goal: “smartly” route information at NET.
- Including noise & link interactions makes the model more general :-)
- ... however more difficult :-(
- We tried to capture in our model some “practical” constraints ...
- ... at least we continue to have fun!