Connectivity in Geometric Graphs: Beyond the Standard Model

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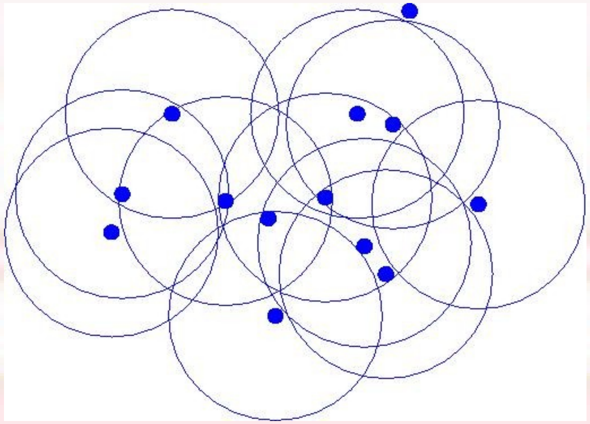
• *The standard continuous percolation model addresses the fundamental question: when a random geometric graph (say, the r-disk graph) on a random* ***iid*** *sample is connected.*

• *Well-known results (Penrose, Gupta-Kumar) establish the connectivity scalings and asymptotic behavior of the connectivity probability.*

$$w_d n r_n^d \approx \log n$$



• *we introduce and discuss the notion of the t-connectivity*

• *we look into the large deviation regimes for t-connectivity problem;*

• *we discuss the influence of the cooperative communications on the connectivity thresholds;*

• *Let A be a plane domain with piece-wise smooth boundary, X an iid size n sample from some density f on A.*

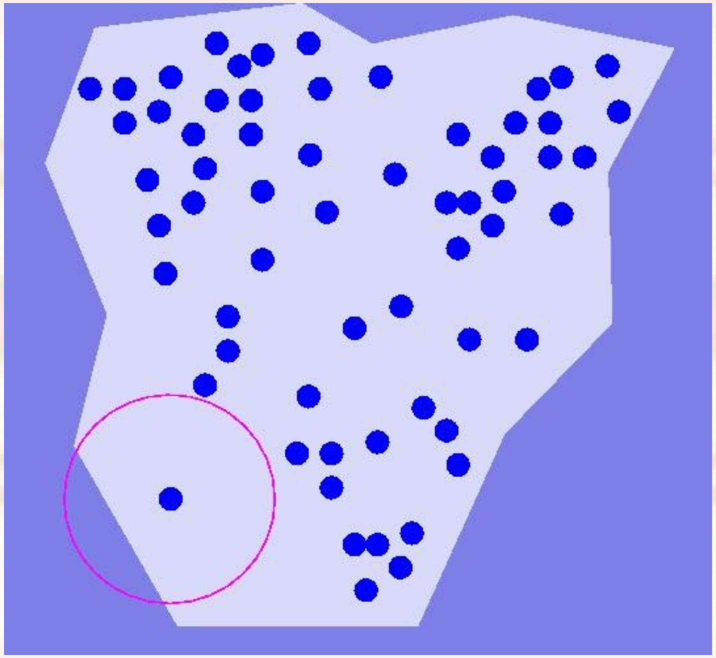
G(X,r) =(E,X) is the graph with vertices set X, and edge (x,y) in E iff |x-y|is at most r.

• *With high probability,*

the loss of connectivity of the graph G(X,r) is caused by singleton nodes

• *Poisson approximation implies the main scaling law* $w_d n r_n^d \approx \log n$

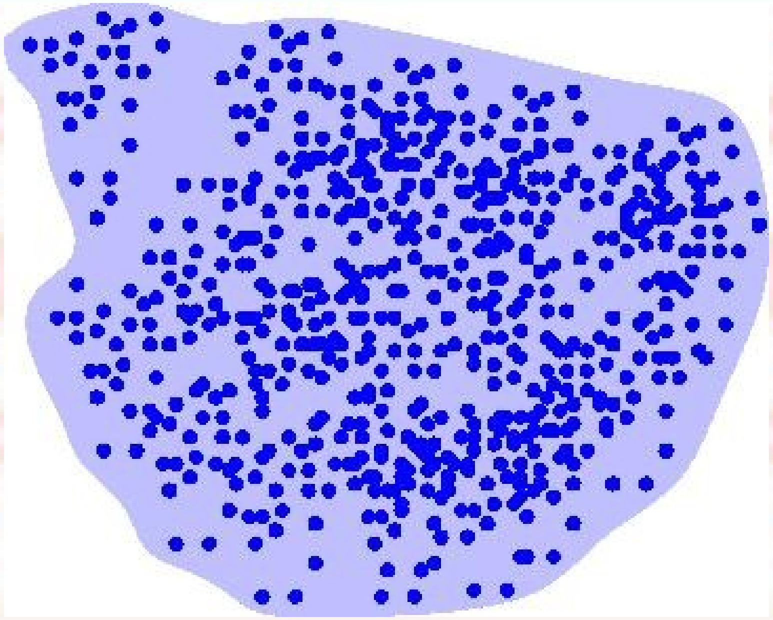
• *Here* $\exp(-w_d n r_n^d)$ *is fighting n*



This result/ intuition is valid in much more general situation:

- *higher dimension*
- *more general domains*
- *non-constant densities*
- *dependent points in the sample*

• *However, often one is less worried about isolated nodes than about losing connectivity to a significant fraction of the network.*



Definition

We say that a graph G=(E,X) is t-disconnected if the largest connected component of G contains less than (1-t)|X| nodes.

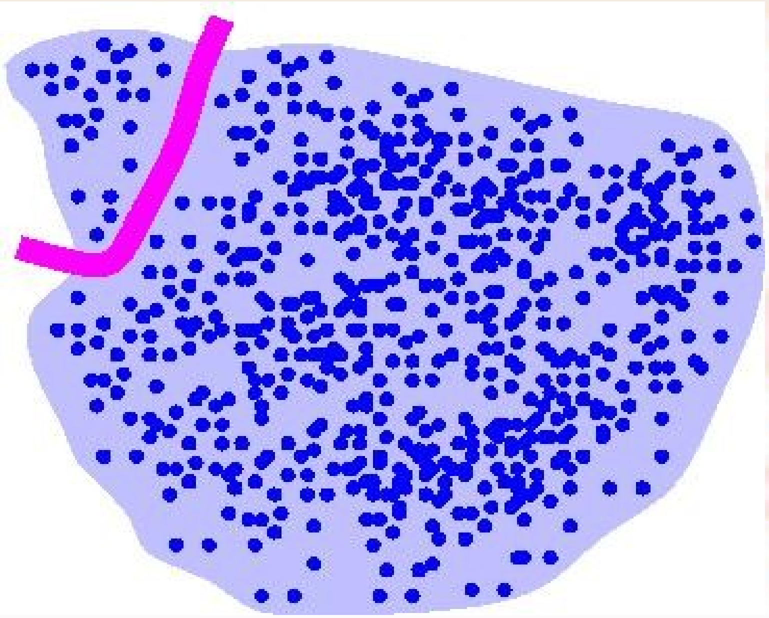
The t-disconnect might occur in a variety of ways:

- *many isolated nodes of size O(1);*
- *many isolated components of size $O(n^{a)})$*
- *...*
- *one large connected component of size ~tn.*

• *For dense random geometric graphs G, t-disconnect is a rare event. Yet its probability is a very important characteristics of the network.*

- *We are concerned with rare event – hence concentrate on the Large Deviation Principle, yielding exponential rate of decay of probability of t-disconnect.*
- *Fix a sequence of radii r_n defining a sequence of*

random geometric graphs $G(X, r_n)$, $|X|=n$.



Assume that the graph $G(X, r_n)$ is dense, i.e.

$$wnr_n^2 \sim \log(n).$$

Let $L(n,t)=\{G(X, r_n)$ has t-disconnect$\}$

Then the following LDP holds:

$$\lim \frac{\log \boldsymbol{P}(L(n,t))}{n\, r_n} = D_A(t)$$

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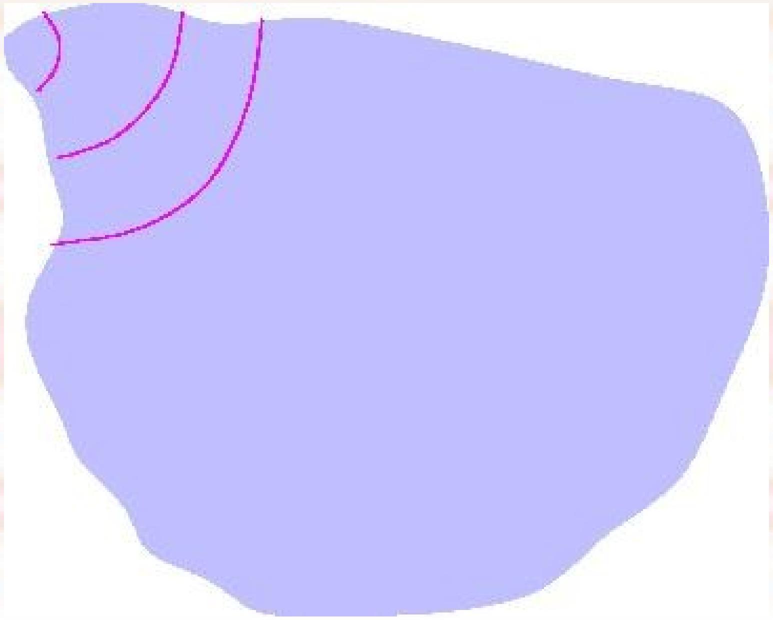
$$\lim \frac{\log \boldsymbol{P}(A(n,t))}{n r_n} = D_A(t)$$

- *The function $D_A(t)$ has a clear geometric meaning: it is the solution of the dual Dido problem (isoperimetric problem):*

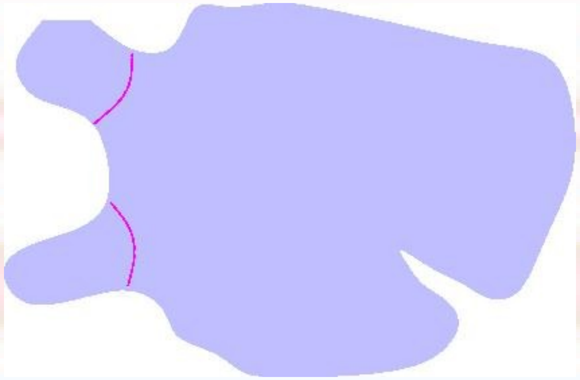
  To bound area t in domain A with a border of minimal length

  Solution: arcs of circles and straight segments

- *In particular, we know where t-disconnect happens. In fact, full Donsker-Varadhan LDP can be established.*
- *This is analogous to Wulff shape theory (describing crystal shapes as solutions to variational problems of minimizing surface interface).*
- *Valid in higher dimensions, for nonuniform samples...*



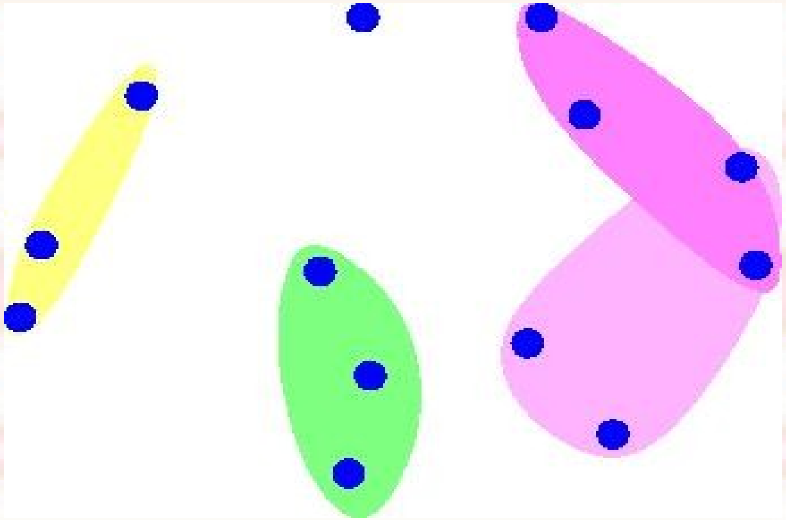
Interface is not necessarily connected

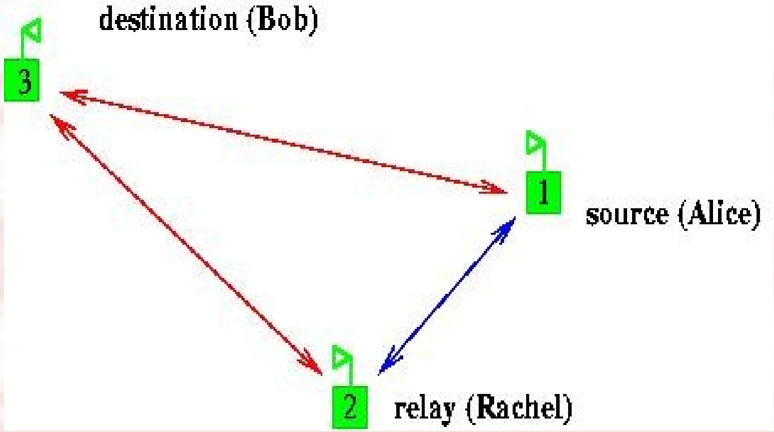


- *Cooperative or assisted communication can improve connectivity characteristics of a random network*

- *In complete generality, the connectivity (ability to pass data with assured minimal rate) is a function of tuples of "close" nodes*

- *Here we address the simplest case: two nodes within direct communication radius r can communicate with a node at distance R>r.*

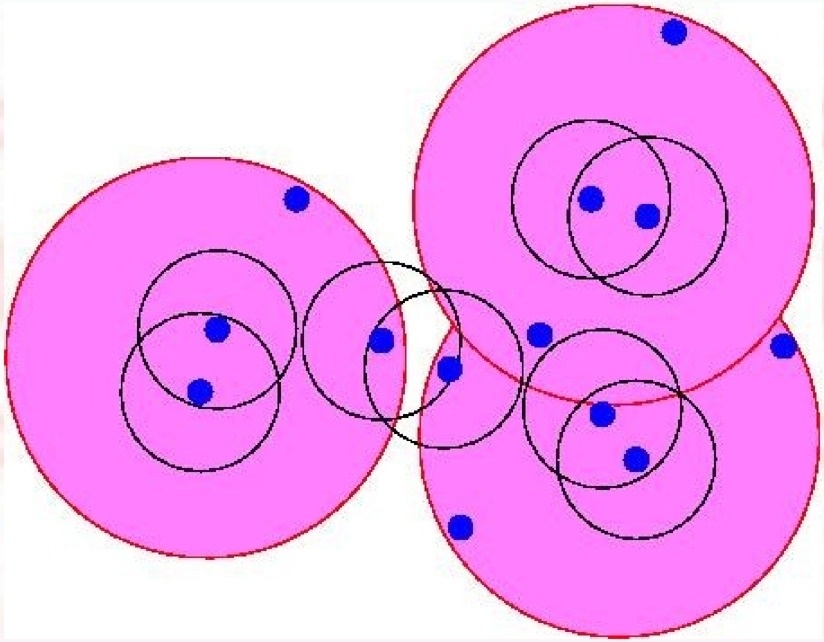




- *The graph G=G(X,r,R) is a supergraph of G(X,r) on the same vertex set X*

- *The edge (i,j) which is not in G(X,r) belongs to G(X,r,R) if and only if there exists a node k such that either*

- *Note that G(X,r,R) is not equal to G(X,R): the range increases only when the communication is assisted*



Main question:can one achieve connectivity with a smaller direct radius using assisted communication mode?

- *Indeed:*

  if

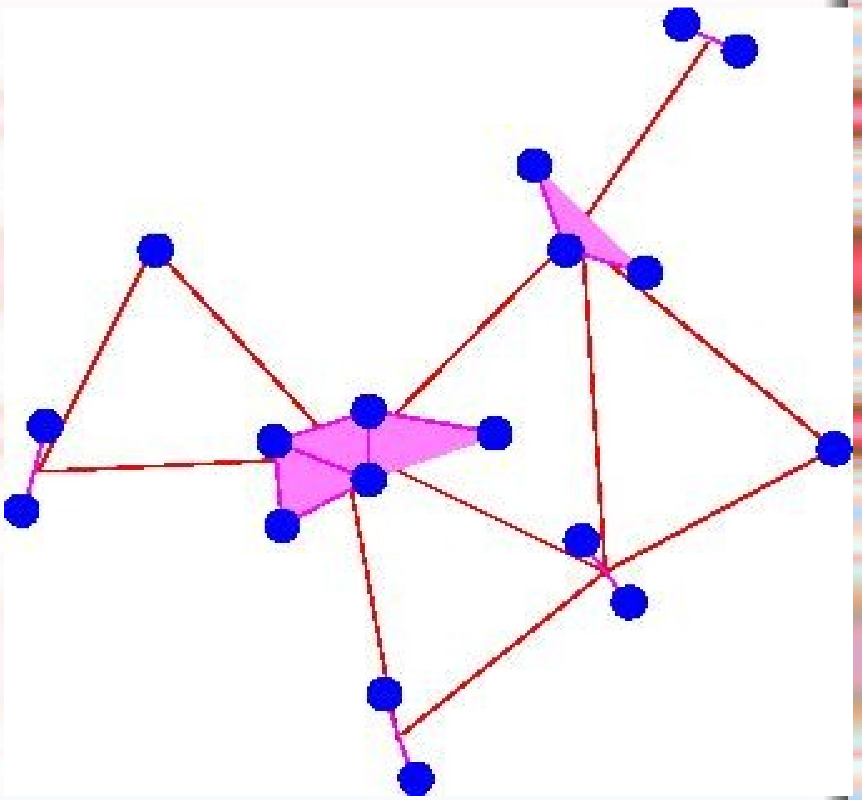
  $$w\,n\,R^2 - \log n \to \infty$$

  and

  $$r \to \infty$$

then G(X,r,R) is connected with high probability

- *In other words, direct communication radius r can grow arbitrarily slowly as a function of n as long as the assisted communication radius R overtakes the standard growth rate*



Assisted communication creates hierarchical network structure: assisted links emerge between clusters spanned by direct communication links

- *thanks !*