Connectivity in Geometric Graphs: Beyond the Standard Model

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• The standard continuous percolation model addresses the fundamental question: when a random geometric graph (say, the $r$-disk graph) on a random iid sample is connected.

• Well-known results (Penrose, Gupta-Kumar) establish the connectivity scalings and asymptotic behavior of the connectivity probability.

\[ w_d n r_n^d \approx \log n \]

• we introduce and discuss the notion of the $t$-connectivity

• we look into the large deviation regimes for $t$-connectivity problem;

• we discuss the influence of the cooperative communications on the connectivity thresholds;
Let $A$ be a plane domain with piece-wise smooth boundary, $X$ an iid size $n$ sample from some density $f$ on $A$.

$G(X,r) = (E,X)$ is the graph with vertices set $X$, and edge $(x,y)$ in $E$ iff $|x-y|$ is at most $r$.

With high probability, the loss of connectivity of the graph $G(X,r)$ is caused by singleton nodes.

Poisson approximation implies the main scaling law $w_d n r_n^d \approx \log n$.

Here $\exp(-w_d n r_n^d)$ is fighting $n$.

This result/intuition is valid in much more general situation:

- higher dimension
- more general domains
- non-constant densities
- dependent points in the sample
• However, often one is less worried about isolated nodes than about losing connectivity to a significant fraction of the network.

**Definition**

We say that a graph $G = (E, X)$ is $t$-disconnected if the largest connected component of $G$ contains less than $(1-t)|X|$ nodes.

The $t$-disconnect might occur in a variety of ways:

- many isolated nodes of size $O(1)$;
- many isolated components of size $O(n^a)$
- ...
- one large connected component of size $\sim n$. 

• For dense random geometric graphs $G$, $t$-disconnect is a rare event. Yet its probability is a very important characteristics of the network.
• We are concerned with rare event – hence concentrate on the Large Deviation Principle, yielding exponential rate of decay of probability of t-disconnect.
• Fix a sequence of radii \( r_n \) defining a sequence of random geometric graphs \( G(X, r_n), |X| = n \).

Assume that the graph \( G(X, r_n) \) is dense, i.e. \( \omega n r_n^2 \sim \log(n) \).
Let \( L(n,t) = \{ G(X, r_n) \text{ has t-disconnect} \} \)
Then the following LDP holds:

\[
\lim \frac{\log P(L(n, t))}{n r_n} = D_A(t)
\]
The function $D_A(t)$ has a clear geometric meaning: it is the solution of the dual Dido problem (isoperimetric problem):

To bound area $t$ in domain $A$ with a border of minimal length

Solution: arcs of circles and straight segments

In particular, we know where $t$-disconnect happens. In fact, full Donsker-Varadhan LDP can be established.

This is analogous to Wulff shape theory (describing crystal shapes as solutions to variational problems of minimizing surface interface).

Valid in higher dimensions, for nonuniform samples...

$$\lim \frac{\log P(A(n,t))}{n r_n} = D_A(t)$$
• Cooperative or assisted communication can improve connectivity characteristics of a random network

• In complete generality, the connectivity (ability to pass data with assured minimal rate) is a function of tuples of “close” nodes

• Here we address the simplest case: two nodes within direct communication radius $r$ can communicate with a node at distance $R > r$. 
• The graph $G=G(X,r,R)$ is a supergraph of $G(X,r)$ on the same vertex set $X$.

• The edge $(i,j)$ which is not in $G(X,r)$ belongs to $G(X,r,R)$ if and only if there exists a node $k$ such that either

• Note that $G(X,r,R)$ is not equal to $G(X,R)$: the range increases only when the communication is assisted.

Main question: can one achieve connectivity with a smaller direct radius using assisted communication mode?
Indeed:

\[
wn R^2 - \log n \rightarrow \infty
\]

and

\[
r \rightarrow \infty
\]

then \( G(X, r, R) \) is connected with high probability

In other words, direct communication radius \( r \) can grow arbitrarily slowly as a function of \( n \) as long as the assisted communication radius \( R \) overtakes the standard growth rate

Assisted communication creates hierarchical network structure: assisted links emerge between clusters spanned by direct communication links
• thanks!