Spectral Analysis
ELEC 532
Assignment 4

Due: Wednesday 23 February 2000

1. Consider the linear prediction of a complex sinusoid by a 1st order predictor. For an ACF given by $r_{xx}[k] = P_1 e^{j2\pi f_1 k}$, find the optimal prediction coefficient α_1 and the minimum prediction error power $\rho_{\min}$. Where are the zeros of the prediction error filter located?

2. **Levinson Algorithm**

   In this problem, we will derive the Levinson algorithm, an efficient recursive scheme for solving the Yule-Walker equations. Our goal is find $\underline{\alpha}_{p+1}, \rho_{p+1}^{\min}$ solving

$$R_{p+1} \begin{bmatrix} 1 \\ \underline{\alpha}_{p+1} \end{bmatrix} = \begin{bmatrix} \rho_{p+1}^{\min} \\ \underline{0}_{p+1} \end{bmatrix} \tag{1}$$

   in terms of the solution $\underline{\alpha}_p, \rho_p^{\min}$ to

$$R_p \begin{bmatrix} 1 \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} \rho_p^{\min} \\ \underline{0}_p \end{bmatrix}.$$

   (a) Define the $p \times p$ exchange (flip) matrix J as the matrix of all zeros except for ones along the NE–SW diagonal. What effect does J have on vectors and matrices? Show that

$$J \begin{bmatrix} 1 \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_p^b \\ 1 \end{bmatrix},$$

   where $\underline{\alpha}_p^b$ is the vector $\underline{\alpha}_p$ written in backwards order. What type of linear prediction problem would these flipped coefficients be natural for?

   Show also that

$$J\, R_p\, J = R_p^*.$$

   (b) Show that R_{p+1} can be decomposed as

$$R_{p+1} = \begin{bmatrix} r_{xx}[0] & \underline{r}_{p+1}^T \\ \underline{r}_{p+1}^* & R_p \end{bmatrix} = \begin{bmatrix} R_p & J\underline{r}_{p+1} \\ (J\underline{r}_{p+1})^H & r_{xx}[0] \end{bmatrix}.$$

(c) **GUESS STEP™**

The Levinson algorithm is a "guess and correct" scheme. Given $\underline{\alpha}_p, \rho_p^{\min}$, we will guess the values of $\underline{\alpha}_{p+1}, \rho_{p+1}^{\min}$. While our guesses will not be correct, they will be sufficiently close that we will be able to correct them to hit right on.

Our guess for $\underline{\alpha}_{p+1}$ is really simple:

$$\begin{bmatrix} 1 \\ \underline{\alpha}_p \\ 0 \end{bmatrix}.$$

(What could be simpler?) However, this guess does not quite work. Show that

$$R_{p+1} \begin{bmatrix} 1 \\ \underline{\alpha}_p \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} \rho_p^{\min} \\ \underline{0}_p \\ \Delta_p \end{bmatrix} \;\neq\; \begin{bmatrix} \rho_{p+1}^{\min} \\ \underline{0}_{p+1} \end{bmatrix}, \tag{2}$$

with

$$\Delta_p^* \;=\; r_{xx}[p+1] \;+\; \sum_{m=1}^{p} \alpha_p^*[m]\, r_{xx}[p+1-m].$$

Equation (2) is almost the same as (1), except for the Δ_p at the bottom.

(d) Using the flip property of R_{p+1}, show that

$$R_{p+1} \begin{bmatrix} 0 \\ (\underline{\alpha}_p^b)^* \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} \Delta_p^* \\ \underline{0}_p \\ \rho_p^{\min} \end{bmatrix}. \tag{3}$$

(e) **CORRECT STEP™**

Both $\rho_p^{\min}$ and Δ_p are known at this point. Therefore, if we add (2) and (3) together in a tricky way, we should be able to cancel out the term at the bottom. Form

$$\begin{bmatrix} 1 \\ \underline{\alpha}_{p+1} \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ \underline{\alpha}_p \\ 0 \end{bmatrix} \;+\; k_{p+1} \begin{bmatrix} 0 \\ (\underline{\alpha}_p^b)^* \\ 1 \end{bmatrix}. \tag{4}$$

Show that choosing

$$k_{p+1} \;=\; -\frac{\Delta_p}{\rho_p^{\min}}$$

in (4) yields the solution to (1). (Thus our corrected guess comprises the correct linear prediction coefficients of order $p+1$.)

(f) Show that we have

$$\rho_{p+1}^{\min} \;=\; \rho_p^{\min}\left(1 - |k_{p+1}|^2\right).$$

(g) **INITIALIZATION**

The above derivation tells us how to form the optimal pair $\underline{\alpha}_{p+1}, \rho_{p+1}^{\min}$ given the optimal pair $\underline{\alpha}_p, \rho_p^{\min}$. Solve the Yule-Walker equations by hand for the case $p = 1$ to find an initialization point for the algorithm.

(h) **SUMMARY**

Write a clear summary of the steps of your Levinson algorithm.

(i) **OPERATION COUNT**

In general, inverting a $p+1 \times p+1$ matrix requires $O(p^3)$ additions and multiplications. Find the order of computational complexity required by Levinson's algorithm to recursively compute the solution to (1).

(j) **EXAMPLE**

Given the autocorrelation sequence

$$\{r_x x[0], r_x x[1], r_x x[2], r_x x[3], r_x x 4[], \} \;=\; \{128, -64, 80, -88, 89\},$$

use Levinson's algorithm to compute the $p = 1, 2, 3, 4$ optimal linear prediction coefficients. What happens to the mean-square error as p increases? Is this reasonable?

In each case, draw the corresponding prediction error filter.

3. With your study group, discuss and synthesize the major *themes* of this week of lectures. Turn in a one page summary of your discussion. You need turn in only one summary, but include the names of all group members. Please do not write up just a "table of contents."