thus making the recursion,

$$\mathcal{M}_i(z) = \mathcal{M}_{i-1}(z^M) + \sum_{k=1}^{M-1} z^{-k}\mathcal{A}_{0,k}(z^M)\mathcal{M}_{i-1}(z^M). \quad (41)$$

This formula makes transparent exactly what happens in a recursive computation of the correlation functions. At every stage i, we have the $\mathcal{Z}$-transform at points $M^{-i}n$ of the appropriate function. In going to the $(i+1)$st stage, the first term $\mathcal{M}_{i-1}(z^M)$ ensures that values at the previous stage are retained at the even points, while the second term interpolates new values at the between points. This computation scheme is essentially the same as the wavelet-based lowpass/bandpass interpolation scheme in [5].

Let us examine Eqn. 38 more closely. Now $\mathcal{H}_l(z)$ has exactly $2N-1$ points and the first $N-1$ points can can be visualized graphically as

$$\overbrace{x \ \ldots \ x}^{M-1 \text{ times}} \ 0 \ \overbrace{x \ \ldots \ x}^{M-1 \text{ times}} \ 0 \ldots \quad (42)$$

where x stands for some non-zero number. Again for any of the cross-correlations $\mathcal{H}_{l,m}$, the first $N-1$ points in the sequence $\mathcal{M}_1(z)$ can be visualized as in Eqn. 42.

Now let us assume that this is true for all $\mathcal{M}_i(z)$, $i = 1, 2, \ldots, J$. We then show that it is true for $\mathcal{M}_{J+1}(z)$, proving by induction that $\mathcal{M}_i(z)$ has for all i, $\frac{N-M}{M}$ alternate zeros. Clearly $\mathcal{M}_J(z^M)$ looks like

$$x \ \overbrace{0 \ldots 0}^{M-1 \text{ times}} \ x \ \overbrace{0 \ldots 0}^{M-1 \text{ times}} \ \ldots \quad (43)$$

Now for any k, sliding the sequence $\mathcal{A}_{0,k}(z)$ reversed in time, against $\mathcal{M}_J(z^M)$ shows that the result is true for $\mathcal{M}_{J+1}(z)$.

Thus at any stage J, there are $\frac{N-M}{M}$ zeros that correspond to zeros of Ψ_l etc., at a spacing of M^{-J+1}. It turns out that all these zeros are new, in the sense that they do *not* correspond zeros at stage $J-1$ and therefore we have Eqn. 37 for all the positive zeros at the M-adics.

Consider the computational complexity. Let K_1 be the length of the sequence $\mathcal{M}_1(z)$ (i.e $K_1 = 2N-1$). The number of points at stage i is related to that at stage $i-1$ by the formula,

$$K_i = M(K_{i-1} - 1) + 2N - 1 = MK_{i-1} + 2N - M - 1 \quad (44)$$

which means

$$K_i = M^{i-1}(K_1 + 2N - M - 1) - (2N - 1 - M). \quad (45)$$

Now each each of the sequences $\mathcal{A}_{0,k}$, is approximately $2N/M$ points long, and since there are $(M-1)$ of them, the computational complexity at stage i, is roughly $2NK_{i-1}$ adds and multiplies. Therefore, the total number of adds/multiplies is roughly

$$\begin{aligned} 2N \sum_{j=1}^{J-1} K_i = & \\ & 2N\left\{\frac{(4N-2-M)(M^{J-1}-1)}{M-1} - 2N(2N-1-M)\right\} \\ \approx & \ 8N^2M^{J-2} - 4N^2 \end{aligned}$$

5. CONCLUSION

In this paper we studied the correlation structures of the scaling function and wavelets associated with a multiplicity M, compactly supported, wavelet tight frames. We show how to efficiently compute the correlation functions approximately at the M-adic rationals. Furthermore, when the tight frame is, in particular, an orthonormal wavelet basis, we show that the approximations involved in the computation of the samples of the correlations cancel in a manner to make the computations exact. For the case of orthonormal wavelet bases, we also try to give a complete description of the zeros at the M-adic rationals. An interesting fact that arises from the analysis is that all the correlation functions possible have infinitely many zeros *in its support*.

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is well known that [4, 3] for a multiplicity M orthonormal wavelet basis the length of the scaling vector is a multiple of M and therefore $N+1$ does not divide M, while N does! Therefore the first (non-integral) point of the form $\frac{k}{M}$, where $\Psi_0(t)$ is zero is $N+1$ and hence $k_1 = (N+1)$. Now let $k_j = (M^j - M)(N-1)/(M-1) + (N+1)$ for all $j = 1, 2, \ldots, J$. Consider $\Psi_0(\frac{k}{M^{J+1}})$. This is related to $\Psi_0(\frac{k-M^J n}{M^J})$ and for fixed k the minimum value of the argument is $\frac{k-(N-1)M^J}{M^J}$ and so if $\frac{k-(N-1)M^J}{M^J} \geq \frac{k_J}{M^J}$

$$\Psi_0\left(\frac{k}{M^{J+1}}\right) = 0.$$

Hence

$$\begin{aligned} k_{J+1} &= M^J(N-1) + k_J \qquad (25)\\ &= M^J(N-1) + (M^J - M)(N-1)/(M-1) + N + 1 \\ &= (M^{J+1} - M)(N-1)/(M-1) + (N+1) \end{aligned}$$

and hence the proof is complete. The proofs for Ψ_l and $\Psi_{l,m}$ follow by the same arguments as for the case Ψ_0. □

The M-adic rationals are defined by the set

$$\mathbf{Q}_n = \{x \in \mathbb{R} \mid M^n x \in \mathbf{Z},\ n \in \mathbb{N}\}. \qquad (26)$$

Now define the following sequence of sets for every non-negative integer n,

$$S_n^+(N,M) = \left\{ x \mid x \in \left[M^{-n}k_n, \frac{N-1}{M-1}\right]\right\} \cap \mathbf{Q}_n \qquad (27)$$

$$S_n(N,M) = S_n^+(N,M) \bigcup S_n^-(N,M) \qquad (28)$$

$$S(N,M) = \bigcup_{i=0}^{\infty} S_i(N,M). \qquad (29)$$

Now we are ready to give our main result.

Theorem 1 *For a multiplicity M, orthonormal wavelet basis with scaling vector of length N,*

$$S(N,M) \bigcup \{1,-1\} \subset Z(\Psi_l) \qquad (30)$$

$$S(N,M) \bigcup \{1,0,-1\} \subset Z(\Psi_{l,m}) \qquad (31)$$

Proof: From Lemma 1 and Eqn. 18 we get for all k

$$\frac{k_n}{M^n} \leq \frac{k}{M^n} \leq (N-1), \qquad (32)$$

$\Psi_l(k/M^n)$ and $\Psi_{l,m}(k/M^n)$ are zero. Therefore from the definition of $S_n^+(N,M)$

$$S_n^+(N,M) \subset Z(\Psi_l), \qquad (33)$$

and since this is true for all positive n, and the correlation functions are symmetric we have

$$S(N,M) \subset Z(\Psi_l). \qquad (34)$$

Similar arguments follow for the zeros of $\Psi_{l,m}$. Notice that the smallest number in $S^+(N)$ is 2. Since $\Psi_l(n) = \delta(n)$, 1 is trivially contained in $Z(\Psi_l)$. Again since $\Psi_{l,m}(n) = 0$, both 0 and 1 are contained in $Z(\Psi_{l,m})$. Hence the result follows. □

A natural question at this point is whether this is a complete description of the zero sets of the correlation functions. For the $\Psi_0(t)$ and $\Psi_{l,m}(t)$, we do believe that this is indeed the case, though we have no proof. One useful fact that may lead to a solution to this problem is the fact that the correlation functions are continuous, and therefore if we have a complete description of the zeros at the M-adics we know all the zeros. As far as the autocorrelations of the wavelets are concerned, it is easy to see even in the multiplicity 2 case, that there are other zeros. From Eqn. 23 we get

$$\sum_{l=0}^{M-1} \Psi_l(t) = M\,\Psi_l(Mt) \qquad (35)$$

For $M = 2$, $\Psi_1(t) = 2\Psi_0(2t) - \Psi_0(t)$. Therefore if $2\Psi_0(2t) = \Psi_0(t)$, then we have a zero of $\Psi_1(t)$. Indeed this is always the case! This is probably the only other zero that arises. Hence we have the following conjecture.

Conjecture 1

$$Z(\Psi_0) = \{-1,1\} \bigcup S(N,M)$$

$$Z(\Psi_{l,m}) = \{-1,0,1\} \bigcup S(N,M)\ l \neq m$$

$$Z(\Psi_l) = T_l \bigcup S(N,M)\ l \neq 0$$

where T_l contains all points that are solutions of the following equation

$$\sum_{i \neq l} \Psi_i(t) - M\,\Psi_0(Mt) = 0. \qquad (36)$$

For example, in the multiplicity 2 case, $T_1 = \{-t_0, t_0\}$ where t_0 satisfies $\Psi_0(t) - 2\Psi_0(2t) = 0$.

4. TRANSFORM DOMAIN INTERPRETATION

The goal of this subsection is to give a recursive formula for the zeros of the correlation functions at the M-adic rationals. One of the consequences will be yet another proof of Theorem. 1. Besides, one gets a nice recursive formula for the zeros of Ψ_l, and $\Psi_{l,m}$. More specifically, for a given N, the elements of $S^+(N)$ are given by the the following recursion.

$$z_n = z_{n-1} + M^{-\lfloor Mn/(N-M)\rfloor} \qquad (37)$$

with $z_0 = 2$. Define

$$\mathcal{M}_i(z) = \mathcal{H}_0(z)\mathcal{M}_{i-1}(z^M) \quad i \geq 2 \qquad (38)$$

with $\mathcal{M}_1(z)$ suitably initialized. Notice that with $\mathcal{M}_1(z) = \mathcal{H}_l(z)$,

$$\mathcal{M}_J(z) = \mathcal{Z}(\Psi_l(M^{-J}n))(z) \qquad (39)$$

Similarly by the initialization $\mathcal{M}_1(z) = \mathcal{H}_{l,m}(z)$ we have $\mathcal{M}_J(z) = \mathcal{Z}(\Psi_{l,m}(M^{-J}n))(z)$. Thus all the correlation functions may be obtained through the *same recursive process with different initializations.* Now decompose $\mathcal{H}_0(z)$ into its polyphase components $\mathcal{A}_{0,k}(z)$

$$\begin{aligned} \mathcal{H}_0(z) &= \mathcal{A}_{0,0}(z^M) + \sum_{k=1}^{M-1} z^{-1}\mathcal{A}_{0,k}(z^M) \qquad (40)\\ &= 1 + \sum_{k=1}^{M-1} z^{-k}\mathcal{A}_{0,k}(z^M) \end{aligned}$$

up to J terms. However, since $H_0(\omega)$ and $H_l(\omega)$ are periodic with period 2π, it should be clear that the truncated product of up to J terms is periodic with a period $2M^J\pi$. Notice that $\hat{\psi}_l(\omega)$ is not periodic. However, the Fourier transform of $\psi_l(M^{-J}n)$ will be periodic. Using this fact, the inverse Fourier transform of the truncated product can be interpreted as the Fourier transform of the samples of $\psi_l(t)$. More precisely,

$$\frac{1}{M^J}\sum_n \psi_l(M^{-J}n)e^{-i\frac{\omega}{M^J}n} \approx \quad (8)$$
$$\left[\frac{1}{\sqrt{M}}H_l\left(\frac{\omega}{M}\right)\right]\prod_{i=2}^{J}\left[\frac{1}{\sqrt{M}}H_0\left(\frac{\omega}{M^i}\right)\right]$$

or equivalently in terms of the $\mathcal{Z}$ transforms

$$\mathcal{Z}(\psi_l(M^{-J}n))(z) \approx M^{J/2}H_l\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}H_0\left(z^{M^i}\right). \quad (9)$$

Therefore the $\mathcal{Z}$-transforms of the correlation functions obtained by the sampled autocorrelation method is given by

$$\mathcal{Z}(\tilde{\Psi}_l(M^{-J}n))(z) \approx \mathcal{H}_l\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_0\left(z^{M^i}\right) \quad (10)$$

and

$$\mathcal{Z}(\tilde{\Psi}_{l,m}(M^{-J}n))(z) \approx \mathcal{H}_{l,m}\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_0\left(z^{M^i}\right) \quad (11)$$

where

$$\mathcal{H}_l(z) = H_l(z)H_l(z^{-1}) \quad (12)$$
$$\mathcal{H}_{l,m}(z) = H_l(z)H_m(z^{-1}). \quad (13)$$

We now show that the approximate correlations obtained above turn out to be the *exact* correlations when the scaling vector h_0 *gives rise to an orthonormal wavelet basis, rather than a wavelet tight frame*. This has the following interesting consequence. The approximation described earlier to compute the samples of the functions $\psi_l(t)$ is good only for large enough J (5 or 6). However, in the case of an orthonormal wavelet basis, even for $J = 1$, the sampled correlations are the actual correlation, even though the samples of $\psi_l(t)$ themselves are not exact! To see this first notice that since for all l $\{\psi_l(t-k)\}$ forms an orthonormal basis,

$$\sum_k \left|\hat{\psi}_l(\omega+2\pi k)\right|^2 = 1. \quad (14)$$

Now the Fourier transform of Ψ_l is given by

$$\hat{\Psi}_l(\omega) = \left|\hat{\psi}_l(\omega)\right|^2 =$$
$$\left[\frac{1}{M}\mathcal{H}_l\left(\frac{\omega}{M}\right)\right]\left\{\prod_{i=2}^{J}\left[\frac{1}{M}\mathcal{H}_0\left(\frac{\omega}{M^i}\right)\right]\right\}\left|\hat{\psi}_0\left(\frac{\omega}{M^J}\right)\right|^2.$$

Therefore by the Sampling Theorem [7],

$$\frac{1}{M^J}\sum_n \Psi_l(M^{-J}n)e^{-i\frac{\omega}{M^J}n} = \sum_k \left|\hat{\psi}_l(\omega+2\pi M^J k)\right|^2 =$$
$$\frac{1}{M}\mathcal{H}_l\left(\frac{\omega}{M}\right)\left\{\prod_{i=2}^{J}\frac{1}{M}\mathcal{H}_0\left(\frac{\omega}{M^i}\right)\right\}\sum_k\left|\hat{\psi}_0\left(\frac{\omega}{M^J}+2\pi k\right)\right|^2$$

and using Eqn. 14 we get

$$\sum_n \Psi_l(M^{-J}n)e^{-i\frac{\omega}{M^J}n} = \mathcal{H}_l\left(\frac{\omega}{M}\right)\prod_{i=2}^{J}\mathcal{H}_0\left(\frac{\omega}{M^i}\right) \quad (15)$$

$$\mathcal{Z}(\Psi_l(M^{-J}n)) = \mathcal{H}_l\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_0\left(z^{M^i}\right). \quad (16)$$

Similarly, it can be shown that,

$$\mathcal{Z}(\Psi_{l,m}(M^{-J}n)) = \mathcal{H}_{l,m}\left(z^{M^{J-1}}\right)\prod_{i=0}^{J-2}\mathcal{H}_0\left(z^{M^i}\right) \quad (17)$$

3. ZEROS OF Ψ_l, $\Psi_{l,m}$

In this section we only consider multiplicity M orthonormal wavelet bases for which the correlation functions can be computed exactly at the M-adic rationals. Let $Z(f)$ denote the zero set of a function f in its support. If the scaling vector h_0 is of length N, then all the wavelet vectors h_l are of length N and without loss of generality the supports of $\psi_l(t)$ can be considered to be in $[0, \frac{N-1}{M-1})$ ([3]) and hence are all of width $\frac{N-1}{M-1}$. Therefore

$$Supp(\Psi)_l = [-\frac{N-1}{M-1}, \frac{N-1}{M-1}] \quad (18)$$

$$Supp(\Psi)_{l,m} = [-\frac{N-1}{M-1}, \frac{N-1}{M-1}]. \quad (19)$$

In order to prove our main result we need the following Lemma.

Lemma 1 *Let k_J be the first positive integer such that for integers $k \geq k_J$*

$$f\left(\frac{k}{M^J}\right) = 0 \quad (20)$$

where f is either $\Psi_l(t)$ or $\Psi_{l,m}(t)$. Then k_J is given by the formula

$$k_J = (N-1)(M^J - M)/(M-1) + (N+1) \quad (21)$$

Proof: Let $\mathbf{h}_l(n)$ and $\mathbf{h}_{l,m}(n)$ be the correlations of the sequences $h_l(n)$. Then, from fact that $h_0(n)$ is a scaling vector, viz. Eqn. 3,

$$\mathbf{h}_l(Mn) = \delta(n) \text{ and } \mathbf{h}_{l,m}(Mn) = 0, \quad (22)$$

and therefore,

$$\Psi_l(t) = \Psi_0(Mt) + \sum_{\substack{n=-(N-1)\\ M\not| n}}^{(N-1)}\mathbf{h}_l(n)\Psi_0(Mt-n) \quad (23)$$

$$\Psi_{l,m}(t) = \sum_{\substack{n=-(N-1)\\ M\not| n}}^{(N-1)}\mathbf{h}_{l,m}(n)\Psi_0(Mt-n). \quad (24)$$

We now use induction to prove the formula for k_J. Consider $\Psi_0(\frac{k}{M})$. This is related to $\Psi_0(k-n)$, and for fixed k the minimum value of the argument is $k-(N-1)$. Since $\Psi_0(n) = \delta(n)$, for all $k > N-1$, $\Psi_0(\frac{k}{M})$ is zero. But it

ON THE CORRELATION STRUCTURE OF MULTIPLICITY M SCALING FUNCTIONS AND WAVELETS

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ABSTRACT

In this paper we study the auto-correlation and cross-correlation structure of the scaling and wavelet functions associated with compactly supported orthonormal wavelet bases. These correlation structures play an important role in both wavelet-based interpolation [5] and in answering the question of *existence* of scale-limited signals [6]. Our investigations into their nature, gives us a fairly complete account of all the zeros of the correlation functions and also give efficient algorithms for their computation. An interesting fact that arises from the analysis is that all the correlations possible have infinitely many zeros in their support.

1. INTRODUCTION

Recently multiplicity M, K-regular, compactly supported, orthonormal and biorthogonal wavelet bases have been constructed [4, 3]. These wavelet bases are extensions of the multiplicity 2, K-regular, orthonormal wavelet bases constructed by I.Daubechies [1]. A multiplicity M, compactly supported, orthonormal wavelet basis is completely determined by a *scaling vector* h_0 and $M-1$ wavelet vectors h_1 through h_{M-1} all of finite length N. The scaling function and wavelets associated with the scaling and wavelet vectors are given respectively by

$$\psi_0(t) = \sqrt{M} \sum_{k=0}^{N-1} h_0(k)\psi_0(Mt-k), \tag{1}$$

$$\psi_l(t) = \sqrt{M} \sum_{k=0}^{N-1} h_l(k)\psi_0(Mt-k) \text{ for } l = 1,\ldots M-1 \tag{2}$$

where $\psi_0(t)$ is the scaling function and $\psi_l(t)$ for $l = 1,\ldots,M-1$ are the $M-1$ wavelets. The scaling and wavelet vectors satisfy the following quadratic and linear constraints that completely characterizes them.

$$\sum_k h_l(k)h_m(k+Mn) = \delta(l-m)\delta(n) \tag{3}$$

$$\sum_k h_l(k) = \sqrt{M}\delta(l) \tag{4}$$

2. CORRELATION STRUCTURE OF SCALING FUNCTION AND WAVELETS

The auto-correlation and cross-correlations of the functions $\psi_l(t)$ are defined by

$$\Psi_l(t) = \int_{\mathbb{R}} \psi_l(\tau)\psi_l(t+\tau)\,d\tau \tag{5}$$

$$\Psi_{l,m}(t) = \int_{\mathbb{R}} \psi_l(\tau)\psi_m(t+\tau)\,d\tau. \tag{6}$$

Since the only general way to construct a multiplicity M wavelet tight frame is by constructing a scaling vector h_0, there is usually no explicit control over the nature of the scaling function or the wavelets. More specifically, there is no analytical method to describe the scaling function and wavelets, and the only alternative is to compute samples of them numerically (approximately). Since the functions themselves cannot be obtained explicitly, it is not surprising that in general it is not possible to obtain the autocorrelations and cross-correlations exactly. However, given the uniform samples of the scaling function and wavelets at a fine enough sampling rate, the correlation functions can be obtained by a step-approximation to the integral in Eqns. 5 and 6. Though the scaling function and wavelet can be computed exactly at the rationals, it turns out that for special rationals, specifically the M-adic rationals, there are efficient ways to compute these samples. The authors are yet unaware of any technique to compute the scaling function and wavelet at arbitrary irrationals. There are two efficient methods to compute the scaling function and wavelet samples at the M-adics. The first method computes the exact values of these functions at the integers by solving a suitable eigenvalue problem, while the other method is an iterative approximation technique [4, 2].

The approximate method (*infinite product method*) has two advantages. Firstly, since it does not involve the solution of a linear set of equations it is more efficient. Secondly, if the wavelet tight frame turns out to be an orthonormal wavelet basis, (which is typical), then the approximate method turns out to be exact!! More specifically, notice that there are two approximation processes involved. Firstly, the integrals for the correlations are being approximated by a step-approximation, and secondly the samples of the functions $\psi_l(t)$ are themselves being approximated. In a nice fashion the two approximation processes – that of obtaining the samples of ψ_l, and that of computing the step approximation cancel each other to give the correlation functions exactly.

Consider the computation of $\psi_l(M^{-J}n)$ for fixed l and J using the *infinite product method*. From Eqns. 1 and 2, by taking the Fourier transform on both sides it is easy to see that

$$\hat{\psi}_l(\omega) = \left[\frac{1}{\sqrt{M}} H_l\left(\frac{\omega}{M}\right)\right] \prod_{i=2}^{\infty} \left[\frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^i}\right)\right]. \tag{7}$$

Now the essential idea in the infinite product method is to approximate the infinite product by a finite product of

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