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Abstract

This paper constructs K-regular M-band orthonormal wavelet bases. K-regularity of the wavelet basis is known to be useful in numerical analysis applications and in image coding using wavelet techniques. Several characterizations of K-regularity and their importance are described. An explicit formula is obtained for all minimal length M-band scaling filters. A new state-space approach to constructing the wavelet filters from the scaling filters is also described. When M-band wavelets are constructed from unitary filter banks they give rise to wavelet tight frames in general (not orthonormal bases). Conditions on the scaling filter so that the wavelet bases obtained from it is orthonormal is also described.

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1 Introduction

In recent years wavelet orthonormal bases have been constructed and studied extensively from both a mathematical and a signal processing point of view [6, 27, 28, 5, 3, 47, 48, 36]. One reason that wavelets are interesting is that they overcome some of the shortcomings of short-time Fourier decompositions [30, 17, 48, 7, 20, 13], by decomposing a signal into channels that have the same bandwidth on a logarithmic scale. Thus high frequency channels have wide bandwidth and low frequency channels have narrow bandwidth. These characteristics are well suited for analysis of low frequency signals mixed with sharp transitions (spikes). The disadvantage, however is that if there are high frequency signals with relatively narrow bandwidth (like a long RF pulse), the decomposition is not well suited. In order to overcome this problem M-band orthonormal wavelet bases have been constructed recently by several authors [17, 52, 23], as a direct generalization of the 2-band wavelets of Daubechies [6]. M-band wavelets help to zoom in onto *narrow* band high frequency components of a signal, while simultaneously having a logarithmic decomposition of frequency channels. Moreover, they give better energy compaction than 2-band wavelets [52].

Central to Daubechies' discovery of compactly supported 2-band wavelets is the lowpass filter of a two channel unitary filter bank with a specified order of regularity. In the 2-band case, the lowpass filter (or *unitary scaling filter*) of shortest length with a given regularity order K is fixed (modulo a spectral factorization). The highpass filter (or *unitary wavelet filter*) is uniquely determined by the lowpass filter from the unitariness of the filter bank. Regularity (equivalently stated as vanishing of the wavelet moments) plays an important role in image processing and numerical analysis. This is especially true for tree-structured decompositions such as the multiresolution analysis of Mallat [30], in which one iterates the filter on the lowpass output. Experiments have shown the Daubechies' wavelet filters to be highly effective for image coding [51].

Unitary filter banks are a special class of multirate filter banks, the theory of which is well understood in the signal processing community [41, 49]. Excellent surveys of this work also appear in [43, 46]. While Daubechies' construction of K-regular scaling filters and associated 2-band wavelet bases did not draw from the theory of unitary filter banks, the M-band wavelet bases constructed in [17] are based on deep results in filter bank theory. From the filter bank approach, however, there is no simple scheme to obtain K-regular, M-band scaling filters. In general one has to solve a set of non-linear equations to numerically obtain the filter impulse responses [17].

This paper makes three main contributions. Firstly, explicit formulas for K-regular M-band scaling filters are obtained. Just as in the 2-band case Daubechies' construction, the shortest length

K-regular M-band scaling filter is fixed modulo a spectral factorization. In the M-band case there are $(M-1)$ unitary wavelet filters, and they are not uniquely determined by the scaling filter (unlike the 2-band case). Secondly, two different approaches to the construction of the $(M-1)$ wavelet filters and associated wavelet bases are described. One of them relies on a state-space characterization of compactly supported wavelet bases with a novel technique for obtaining the unitary wavelet filters; the other uses the factorization approach in [17]. The wavelets so constructed in general give rise only to wavelet tight frames (not orthonormal bases). Thirdly, this paper gives a set of necessary and sufficient condition on the M-band scaling filter for it to generate an orthonormal wavelet basis. The conditions are very similar to those obtained by Cohen [4, 8] and Lawton [27] for 2-band wavelets. This distinction between a tight frame and an orthonormal basis is quite subtle (and technical) and has been overlooked in the signal processing community.

The organization of the paper is as follows. Section 2 is a tutorial overview of multirate filter bank theory, wavelet theory and their interrelationship. Section 3 identifies and explores several equivalent notions of wavelet regularity in the M-band case; the most useful one for this paper will be based on flatness of magnitude response of the scaling filter. In section 4 this notion of flatness is used to derive the general K-regular M-band unitary scaling filter of shortest length. Section 5 describes several approaches to the construction of wavelet filters. Section 6 characterizes scaling filters that give rise to orthonormal wavelet bases.

All through this paper by the standard abuse of notation the $\mathcal{Z}$-transform and Fourier transform of a sequence $h(n)$ will be denoted by $H(z)$ and $H(\omega)$ respectively.

2 An Overview of M-band Wavelet Theory

There is a close relationship between FIR perfect reconstruction filter banks and compactly supported wavelet bases in the 2-band case (as well as the general M-band case) [6]. M-band wavelets were initially constructed by exploiting this connection [17, 52, 38]. This section gives a tutorial overview of this connection between perfect reconstruction filter banks and M-band wavelets.

2.1 Perfect Reconstruction Filter Banks

The structure of the classical one-dimensional filter bank problem is given in Fig. 1. The filter bank problem involves the design of the *real* coefficient *realizable* (i.e., FIR or causal stable IIR) filters $h_i(n)$ and $g_i(n)$, with the following goals: *Perfect Reconstruction* (i.e., $y(n) = x(n)$), and approximation of ideal frequency responses (see Fig. 2) [37, 41, 42, 45, 49, 46, 31, 44]. Closely

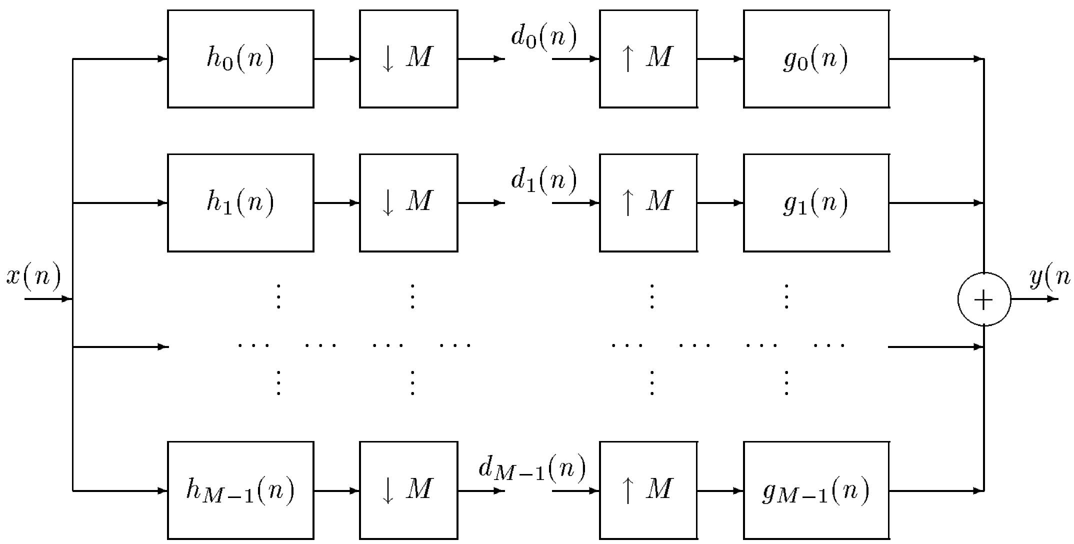

Figure 1: An M-channel Filter Bank

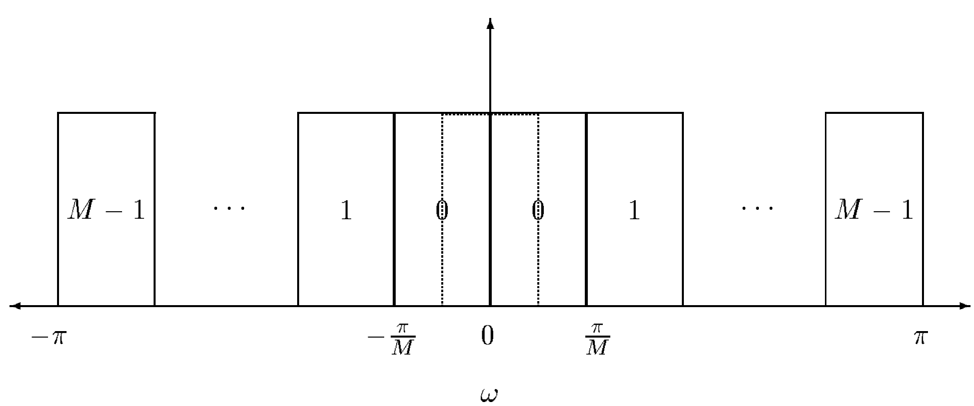

Figure 2: Ideal Frequency Responses in an M-channel Filter Bank

related to the filter bank problem is the transmultiplexer problem (*dual* of the filter bank problem) [49]. A transmultiplexer is a device for converting time-domain-multiplexed (TDM) signals to frequency-domain-multiplexed signals (FDM). The basic structure of a transmultiplexer is shown in Fig. 3. The transmultiplexer problem is to design filters such that perfect reconstruction is guaranteed (i.e., for all i, $x_i(n) = y_i(n)$) and the filter responses approximate Fig 2.

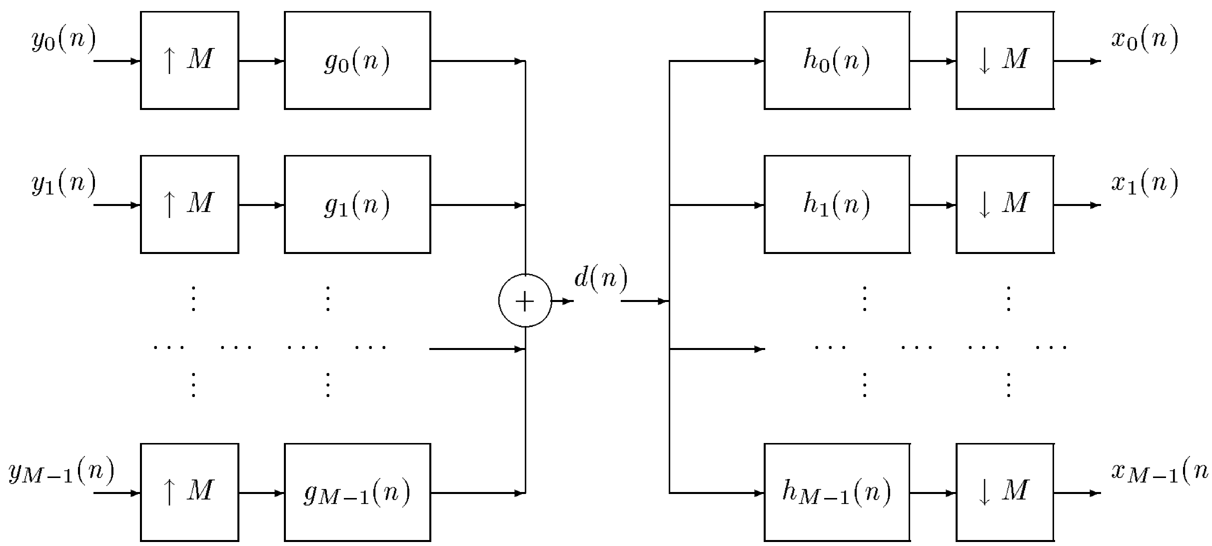

Figure 3: An M-channel Transmultiplexer

For PR the filters h_i and g_i have to satisfy a set of *algebraic* conditions [44]. Let $\mathcal{L}(M) = \{\ldots, -2M, -M, 0, M, 2M, \ldots\}$ and $\mathcal{R}(M) = \{0, 1 \ldots, M-1\}$. $\mathcal{L}(M)$ is the lattice generated by M and $\mathcal{R}(M)$ is the set of *representatives* of $\mathcal{L}(M)$. For any sequence $x(n)$ one can define the polyphase representation (see [44, 2, 15]) with respect to $\mathcal{R}(M)$ as follows:

$$X(z) = \sum_{k \in \mathcal{R}(M)} z^k X_k(z^M). \tag{1}$$

$-\mathcal{R}(M)$ is also a set of representatives, and the polyphase representation of $x(n)$ with respect to $-\mathcal{R}(M)$ will be called *the dual polyphase representation* of $x(n)$.For $k \in \mathcal{R}(M)$, let $X_k(z)$, $Y_k(z)$ and $G_{i,k}(z)$ be the components of the polyphase representations of $X(z)$, $Y(z)$, and $G_i(z)$ respectively with respect to $\mathcal{R}(M)$. Let $H_{i,k}(z)$ be the components of the *dual* polyphase representation of $H_i(z)$

with respect to $\mathcal{R}(M)$:

$$X(z) = \sum_{k \in \mathcal{R}(M)} z^k X_k(z^M) \quad \text{and} \quad Y(z) = \sum_{k \in \mathcal{R}(M)} z^k Y_k(z^M) \tag{2}$$

$$H_i(z) = \sum_{k \in \mathcal{R}(M)} z^{-k} H_{i,k}(z^M); G_i(z) = \sum_{k \in \mathcal{R}(M)} z^k G_{i,k}(z^M). \tag{3}$$

Now define the polyphase component matrices $\mathbf{H}(z)$ and $\mathbf{G}(z)$ as follows:

$$(\mathbf{H}(z))_{i,k} = H_{i,k}(z); (\mathbf{G}(z))_{i,k} = G_{i,k}(z). \tag{4}$$

Fact 1 *A filter bank has the PR property iff*

1. $\mathbf{G}^T(z)\mathbf{H}(z) = I$ *or equivalently*

$$\sum_i \sum_n h_i(Mn + n_1) g_i(-Mn - n_2) = \delta(n_1 - n_2). \tag{5}$$

2.

A transmultiplexer has the PR property iff

1. $\mathbf{H}(z)\mathbf{G}^T(z) = I$, *or equivalently*

2.

$$\sum_n h_i(n) g_j(-Ml - n) = \delta(l)\delta(i - j). \tag{6}$$

When the number of channels is equal to the downsampling factor M, a filter bank is PR iff the corresponding transmultiplexer is PR.

For the purposes of this paper the number of channels M will be equal to the downsampling factor and hence we do not have to make a distinction between the filter bank and transmultiplexer PR properties.

Unitary filter banks are a special class of PR filter banks where the synthesis filters are determined by the analysis filters as follows: $g_i(n) = h_i(-n)$. In this case $\mathbf{G}(z) = \mathbf{H}(z^{-1})$ and therefore the PR property becomes $\mathbf{H}^T(z^{-1})\mathbf{H}(z) = I$. In other words, on the unit circle ($z = e^{\imath\omega}$) $\mathbf{H}(z)$ is unitary (hence the name unitary filter banks - unitary filter banks are also known as *paraunitary* filter banks [44, 12]). A filter bank is unitary iff

$$\sum_i \sum_n h_i(Mn + n_1) h_i(Mn + n_2) = \delta(n_1 - n_2). \tag{7}$$

and a transmultiplexer is unitary iff

$$\sum_{n} h_i(n) h_j(Ml+n) = \delta(l)\delta(i-j). \tag{8}$$

Eqn. 8 is also know in the DSP literature as the Nyquist(M) condition [44]. Moreover, it is also the orthogonality conditions found in the literature on lapped orthogonal transforms [31]. One can also write down the expression for unitariness of a transmultiplexer in the frequency domain

$$\frac{1}{M}\sum_{k=0}^{M-1} H_i\left(\frac{\omega-2\pi k}{M}\right) H_j^{\star}\left(\frac{\omega-2\pi k}{M}\right) = \delta(i-j). \tag{9}$$

Notice that $\mathbf{H}(z)\mathbf{H}^T(z^{-1}) = I$ is equivalent to

$$[\downarrow M]\left\{H_i(z)H_j(z^{-1})\right\} = \sum_{k\in\mathcal{R}(M)} H_{i,k}(z)H_{j,k}(z^{-1}) = \delta(i-j), \tag{10}$$

where $[\downarrow M]$ denotes the downsampling operator correponding to a sampling rate change of M.

The class of FIR unitary filter banks are very important because they can be completely parameterized, are easy to implement and there are no questions of stability to be addressed. Moreover, they can be used to construct orthonormal M-band wavelet bases. If the filters are FIR then (by shifting the filters if necessary) $\mathbf{H}(z)$ is a polynomial in z^{-1}, say of degree $(K-1)$. Then the filters can be at most of length MK.

Fact 2 *Every unitary polynomial matrix* $\mathbf{H}(z)$ *of (polynomial) degree* $(K-1)$ *can be uniquely factored in the form ([18])*

$$\mathbf{H}(z) = \prod_{i=K-1}^{1}\left[I - P_i + z^{-1}P_i\right] V_0 \tag{11}$$

where P_i *are projection matrices of rank* δ_i *and* V_0 *is a constant unitary matrix.*

Every rank n projection matrix P can be written (non-uniquely) in the form $w_1 w_1^T + \ldots + w_n w_n^T$, where w_i are unit norm M-vectors that are mutually orthogonal. Therefore if $L = \sum_{i=1}^{K-1}\delta_i$, one has the (non-unique) Householder factorization ([42])

$$\mathbf{H}(z) = \left[\prod_{i=L-1}^{1}\left[I - v_i v_i^T + z^{-1} v_i v_i^T\right]\right] V_0. \tag{12}$$

L is the McMillan degree of $\mathbf{H}(z)$ [44]. The unit M-vectors v_i are known as Householder parameters, and each is determined by $(M-1)$ scalar parameters. Moreover the unitary matrix V_0 is determined by $\binom{M}{2}$ parameters. Therefore a polyphase matrix $\mathbf{H}(z)$ of polynomial degree $(K-1)$ is completely determined by $\binom{M}{2} + (M-1)(L-1)$ parameters where L is the McMillan degree of $\mathbf{H}(z)$. Factorization of polynomial matrices unitary on the unit circle is a direct consequence of classical results in network theory [1, 44].

Consider the filter h_0 of length N, $M(K-1) < N \leq MK$. Then the degree $(K-1)$ polynomial vector $\begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) & \dots & H_{0,M-1}(z) \end{bmatrix}$ has the following characterization [44]:

Fact 3 *Every polynomial vector of (polynomial) degree $(K-1)$ is uniquely determined by $(K-1)$ projection matrices P_i, $i \in \{1, \dots, K-1\}$, each of rank one (i.e., $P_i = v_i v_i^T$) and the vector v_0*

$$\begin{bmatrix} H_{0,0}(z) \\ H_{0,1}(z) \\ \vdots \\ H_{0,M-1}(z) \end{bmatrix} = \left[\prod_{i=1}^{K-1} \left[I - v_i v_i^T + z^{-1} v_i v_i^T \right] \right] v_0, \tag{13}$$

The McMillan degree of this vector polynomial is precisely $(K-1)$. Therefore, the McMillan degree of any one filter in an M-channel filter bank with filters of length MK is always $K-1$. However, the McMillan degree of $\mathbf{H}(z)$ could be $L \geq K$.

In summary, while $\mathbf{H}(z)$ is determined by $\binom{M}{2} + (M-1)(K-1)$ parameters, $H_0(z)$ is determined by $(M-1) + (M-1)(K-1) = (M-1)K$ parameters.

2.2 M-band Wavelets

Under some conditions there is a relationship between general PR filter banks and wavelet frames (biorthogonal bases). In this paper we are interested only in the relationship between unitary filter banks and wavelet tight frames (orthonormal bases) - however, the general relationship will be tabulated at the end of this section. One approach to construct M-band wavelets would be to start with a multiresolution analysis (MRA) as in the 2-band case ([6, 48, 29, 8]) with a scaling factor of M. With this approach one first constructs the scaling filter and then the wavelet filters and wavelets.

Definition 1 (Unitary Scaling Filter) *A unitary scaling filter is a sequence $h_0(n)$ that satisfies the following linear and quadratic constraints*

$$\sum_k h_0(k) h_0(k+Ml) = \delta(l) \quad \sum_k h_0(k) = \sqrt{M} \tag{14}$$

The quadratic condition is precisely that satisfied by the lowpass filter in a unitary filter bank (see Eqn. 6). Therefore, if $H_{0,k}(z)$ denotes the polyphase components of $H_0(z)$, with respect to $\mathcal{R}(M)$, then from Eqn. 10

$$[\downarrow M]\, H_0(z)H_0(z^{-1}) = \sum_{k\in\mathcal{R}(M)} H_{0,k}(z)H_{0,k}(z^{-1}) = 1. \tag{15}$$

The polyphase components of the scaling filter form a polynomial vector that is unitary on the unit circle. Using this and Fact 3 all finite length unitary scaling filters can be parametrized [17, 52]. One can show that linear constraint (in Eqn. 14) is equivalent to the vector v_0 in Eqn. 13 having all its entries $1/\sqrt{M}$. In other words every unitary scaling filter of length $N = MK$ (appending a few zeros *if necessary*) can be parametrized as follows:

$$\begin{bmatrix} H_{0,0}(z) \\ H_{0,1}(z) \\ \vdots \\ H_{0,M-1}(z) \end{bmatrix} = \frac{1}{\sqrt{M}} \prod_{k=K-1}^{1} \left[I - v_i v_i^T + z^{-1} v_i v_i^T\right] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{16}$$

Definition 2 (Unitary Wavelet Filters) *Given a unitary scaling filter we define unitary wavelet filters to be all possible choices of* h_i, $i \in \{1, \ldots, M-1\}$ *such that the scaling filter and the wavelet filters together are filters of a unitary filter bank.*

For any given unitary scaling filter the corresponding wavelet filters are not unique even if they are all of the same length $N = MK$. *One way* to generate wavelet filters is to *unitarily* complete v_0 in Eqn. 16 or Eqn. 13 (i.e., append orthogonal columns to it) to give an orthogonal matrix V_0. This can be done by a Gram-Schmidt process in $\binom{M-1}{2}$ ways since we are adding exactly $(M-1)$ orthogonal columns. This is the only process of construction of unitary wavelet filters that has been discussed in the literature [17, 52] and is essentially equivalent to [23]. However, this overlooks the crucial fact that *there do exist other choices of wavelet filters with the same length* $N = MK$. To see this, let $\mathbf{H}(z)$ be the polyphase matrix corresponding to the scaling and (one choice of) wavelet filters of length $N = MK$. Then from Fact 2 we have $(K-1)$ projection matrices and a constant matrix V_0 determining these filters. The linear constraint, $\sum_k h_0(k) = \sqrt{M}$, is equivalent to the first row of V_0 having all its entries $\frac{1}{\sqrt{M}}$. If the rank of any one projection matrix is greater than one, the McMillan degree $(L-1)$ is *greater than* the polynomial degree $(K-1)$. However, by the unitary completion process described one can only construct those $\mathbf{H}(z)$ that have McMillan degree equal to polynomial degree (i.e., $L = K$). In this paper we also introduce a state-space approach to constructing the wavelet filters. This method is useful when unitary scaling filters are constructed by techniques that do not rely on Eqn. 13 (as in the K-regular case).

Given the scaling and wavelet filters one constructs the scaling function which is the solution to the following two-scale difference equation that involves only the scaling filter.

$$\psi_0(t) = \sqrt{M} \sum_{k=0}^{N-1} h_0(k)\psi_0(Mt-k) \stackrel{\text{def}}{=} T_0\psi_0(t). \tag{17}$$

In order for a solution in $L^1(\mathbb{R})$ to exist it is necessary that the linear constraint $\sum_k h_0(k) = \sqrt{M}$ is satisfied. In fact for $N < \infty$ this equation, *the scaling recursion*, always has solution in $L^2(\mathbb{R})$ [6, 28]. Moreover, Daubechies and Lagarias [9, 24] prove the existence of a *unique* solution to the scaling recursion in $L^1(\mathbf{R}) \bigcap L^2(\mathbf{R})$. In any case, the unique solution can be constructed by the following process due to Daubechies [6]. One starts with any integrable function and applies the operator T_0 to it recursively. This process converges *weakly* in $L^2(\mathbb{R})$ to the scaling function. This convergence can also be seen in the Fourier transform domain where one has the following infinite product representation:

$$\hat{\psi}_0(\omega) = \prod_{j=1}^{\infty} \left[\frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^j}\right)\right]. \tag{18}$$

This convergence is uniform on compact subsets to the Fourier transform $\hat{\psi}_0(\omega)$. Moreover, one can also show that the scaling function is compactly supported in $\left[0, \frac{N-1}{M-1}\right]$ by standard Paley-Wiener arguments [6].

Given the scaling function, one defines the wavelets, one for each unitary wavelet filter as follows:

$$\psi_i(t) = \sqrt{M} \sum_k h_i(k)\psi_0(Mt-k) \quad \text{ for } i \in \{1, \ldots, M-1\} \tag{19}$$

A fundamental property of the M-band wavelets so constructed is that their translates and dilates by powers of M form a tight frame for $L^2(\mathbb{R})$. A proof of this fact in the M-band case may be found in [17]. The corresponding proof in the 2-band case is due to Lawton [28]. For $i \in \{0, \ldots, M-1\}$ define the following family of functions.

$$\psi_{i,j,k}(t) = M^{j/2}\psi_i(M^j t - k). \tag{20}$$

Fact 4 (M-band Tight Frames Theorem) *The function $\{\psi_{i,j,k}\}$ form an M-band wavelet tight frame for $L^2(\mathbb{R})$. That is, for all $f(t) \in L^2(\mathbb{R})$*

$$f(t) = \sum_k \langle f, \psi_{0,0,k}(t)\rangle \psi_{0,0,k}(t) + \sum_{i=1}^{M-1} \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{i,j,k}(t)\rangle \psi_{i,j,k}(t) \tag{21a}$$

$$= \sum_{i=1}^{M-1} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{i,j,k}(t) \rangle \, \psi_{i,j,k}(t). \tag{21b}$$

Tight frames are a generalization of orthonormal bases. The concept of frames is originally due to Duffin and Schaeffer [11] and an excellent treatise on frames is [50]. More recently the theory of frames in the wavelet context can be found in [7]. In finite dimensions given a non-minimal set of vectors one can always throw out some of them to get a basis. In infinite dimensions, given a complete but non-minimal (redundant) set of functions one *cannot* always throw out some of them and obtain a basis (minimal set). This is precisely why one does not talk about frames in finite dimensions. Frames are generalizations of biorthogonal bases and tight frames are generalizations of orthogonal bases, both obtained by giving up minimality. Though the notion of frames is introduced using the concept of frame bounds, for our purposes it suffices to know that a tight frame is any set of distinguished functions such that any function $f(t)$ can be expressed as a linear combination of these functions with weights given by the inner products of $f(t)$ with the corresponding function as in Eqn. 21.

In summary given a unitary scaling filter, one constructs an unique unitary scaling function, $(M-1)$ wavelet filters and associated wavelet functions which give rise to a wavelet tight frame. None of the conditions so far (unitariness of the scaling filter, etc.) is sufficient to ensure that the wavelet basis function $\{\psi_{i,j,k}\}$ form an orthonormal system. Section 6 gives necessary and sufficient conditions on the scaling filter so that they may give rise to an orthonormal basis. Assume for the moment that the scaling function and its integer translates and the wavelets and their integer translates are orthonormal. Then from Eqn. 17 and Eqn. 19 it follows that

$$\delta(i-j)\delta(l) = \int \psi_i(t)\psi_j(t-l)\, dt = \sum_n h_i(n) h_j(n+Ml). \tag{22}$$

In other words, orthonormality of the wavelet basis implies the unitariness of the filter bank associated with the scaling and wavelet filters. However, the converse is not true.

M-band wavelets also give rise to a multiresolution analysis for $L^2(\mathbb{R})$. If we define the spaces $W_{i,j} = Span\,\{\psi_{i,j,k}\}$ then it can be shown that ([17])

$$\{0\} \subset \ldots W_{0,-1} \subset W_{0,0} \subset W_{0,1} \ldots \subset L^2(\mathbb{R}) \tag{23}$$

and

$$W_{0,j+1} = \bigoplus_{i=0}^{M-1} W_{i,j} \tag{24}$$

3 Regularity of M-band Wavelets

Definition 3 (K-Regular Scaling Filter) *An M-band unitary scaling filter is said to be K-regular if it has a polynomial factor of the form $P^K(z)$, with $P(z) = (1 + z^{-1} + \ldots + z^{-(M-1)})/M$ for maximal possible K. That is,*

$$H_0(z) = \left[\frac{1 + z^{-1} + \ldots + z^{-(M-1)}}{M}\right]^K Q(z) = \frac{1}{M^K z^{MK-1}} \left[\frac{z^M - 1}{z - 1}\right]^K Q(z). \qquad (25)$$

If a scaling filter is K-regular, $H_0(z)$ and its first $(K-1)$ derivatives vanish for $z = e^{\imath 2\pi k/M}, k \in \{1, 2, \ldots, M-1\}$. This is equivalent to a set of $(M-1)(K-1)$ complex linear constraints on the scaling filter. Since the scaling filter is assumed to have real coefficients the zeros occur in complex conjugate pairs. Hence the set of $(M-1)(K-1)$ complex constraints reduce to $(M-1)(K-1)$ real linear constraints on the scaling filter.

It also follows from Definition 3 that every unitary scaling filter is 1-regular. Indeed from the unitariness condition in the Fourier domain (see Eqn. 9), it is clear that $H_0(z)$ vanishes for $z = e^{\imath 2\pi k/M}, k \in \{1, 2, \ldots, M-1\}$. The scaling function and wavelets associated with K-regular scaling filters will be called K-regular scaling function and wavelets respectively.

K-regularity has a number of equivalent characterizations, each of which shows how regularity plays an important role in applications. K-regularity was used by Daubechies in the 2-band case in order to ensure that the scaling filter gave rise to a 2-band ON wavelet basis (not a WTF). Moreover, she also showed that the regularity of the scaling function (measured by the number of continuous derivatives it has - or equivalently its Hölder exponent) increases linearly with the K, the regularity of the scaling filter. If the scaling function is K times differentiable *it is necessary* that the scaling filter is $(K-1)$-regular. K-regularity is equivalent to saying that all polynomials of degree $(K-1)$ are contained in $W_{0,j}$ for all j. This coupled with the compact support of the scaling functions (and wavelets) implies that K-regular scaling functions can be used to capture local polynomial behavior. This feature of K-regular scaling filters is particularly useful in image processing applications [51]. K-regularity is also useful in numerical analysis applications [26], where one tries to approximate operators in wavelet bases. In these applications the regularity K of the scaling filter is a measure of the approximation order. From a purely signal processing point of view K-regularity says that the magnitude squared Fourier transform of the scaling filter is flat of order $2K$ at zero frequency. In fact, the explicit formulas in Section 4 correspond to unitary scaling filters with a maximally flat frequency response at the origin.

The moments of h_i and $\psi_i(t)$, and the partial moments of h_0 are defined respectively as follows:

$$\mu(i,k) = \int t^k \psi_i(t)\,dt \quad , m(i,k) = \sum_n n^k h_i(n) \qquad \text{and} \quad \eta(k,l) = \sum_n (Mn+l)^k h_0(Mn+l), \quad (26)$$

so that $m(0,k) = \sum_{l=0}^{M-1} \eta(k,l)$.

Theorem 1 (Equivalent Characterizations of K-regularity) *A unitary scaling filter is K-regular iff*

1. *The frequency response of the scaling filter has a zero of order K at the M^{th} roots of unity.*
2. *The partial moments up to order K of the scaling filter are equal.*
3. *The magnitude-squared frequency response of the scaling filter is flat of order $2K$ at $\omega = 0$.*
4. *All polynomial sequences up to degree $(K-1)$ can be expressed as a linear combination of M-shifts of the scaling filter.*
5. *All moments up to order $(K-1)$ of the wavelet filters vanish.*
6. *All moments up to order $(K-1)$ of the wavelets vanish.*
7. *Polynomials of degree $(K-1)$ or less are contained in $W_{0,j}$ for all j.*

Proof: From Eqn. 25

$$H_0(\omega) = e^{-\imath(M-1)K\omega/2} \left(\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right)^K Q(\omega),$$

and therefore for small ω, $H_0\left(\omega + \frac{2\pi k}{M}\right) = O(\omega^K)$, $k \in \{1,2,\ldots,M-1\}$, implying that the derivatives up to order $(K-1)$ vanish at the roots of unity. Equivalently for $k \in \{1,\ldots,M-1\}$, $i \in \{0,\ldots,K-1\}$

$$\begin{aligned} 0 = \left[\frac{d^i}{d\omega^i} H_0(\omega)\right]_{\omega=2\pi k/M} &= \sum_n (-\imath n)^i \, h_0(n) e^{-\imath \frac{2\pi k n}{M}} \\ \Rightarrow \sum_{l=0}^{M-1} \left[\sum_n h_0(Mn+l)(Mn+l)^i\right] e^{-\imath\frac{2\pi k l}{M}} &= \sum_{l=0}^{M-1} \eta_{i,l} e^{-\imath \frac{2\pi k l}{M}} = 0 \\ \Rightarrow \eta_{i,l} \quad &\text{is a constant independent of } l. \end{aligned}$$

Because of the unitariness of h_0 one also has for small ω

$$|H_0(\omega)|^2 = M - \sum_{k=1}^{M-1} \left| H_0(\frac{\omega + 2\pi k}{M}) \right|^2 = M - O(\omega^{2K}).$$

It follows immediately that for $k \leq K-1$, one can express n^k as a linear combination of $h_0(Ml+n)$:

$$n^k = \sum_l \alpha_{i,l} h_0(Ml+n).$$

As a consequence of this representation, the moments of the wavelet filters vanish up to order $(K-1)$ since

$$m(i,k) = \sum_n n^k h_i(n) = \sum_l \alpha_{i,l} \left[\sum_n h_i(n) h_0(Ml+n) \right] = 0.$$

Now this implies that $\mu(i,k) = 0$ since (from Eqn. 19) they are related to $m(i,k)$ as

$$\mu(i,k) = \frac{1}{M^{k+\frac{1}{2}}} \sum_{j=0}^{k} \binom{k}{j} m(i,j) \mu(0,k-j).$$

Since the wavelets are compactly supported, the form a basis for $L^2_{\text{loc}}(\mathbb{R})$ and therefore for $k \in \{0, \ldots, K-1\}$,

$$t^k = \sum_k \left\langle t^k, \psi_{0,J,k}(t) \right\rangle \psi_{0,J,k}(t) + \sum_{i=1}^{M-1} \sum_{j=J+1}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{i,j,k}(t) \rangle \psi_{i,j,k}(t)$$

$$\Rightarrow \quad t^k = \sum_k \left\langle t^k, \psi_{0,J,k}(t) \right\rangle \psi_{0,J,k}(t).$$

Therefore polynomials of degree $(K-1)$ can be effectively expressed as linear combinations of $\{\psi_{0,j,k}\}$ for fixed j (one might loosely say $W_{0,j}$ contains polynomials of degree $(K-1)$, even though polynomials are not in $L^2(\mathbb{R}) \supset W_{0,j}$). □

3.1 K-regularity and Regularity of Scaling Functions/Wavelets

The precise relationship between K-regularity of the scaling filter and the smoothess of the scaling functions and wavelets is unknown even in the 2-band case. However, using the techniques in [6] it is easy to show that if $Q(\omega)$ is bounded above by an appropriate constant, then the regularity of the scaling function can be estimated. It can be shown that (see [39] for details)

$$\left| \hat{\psi}_0(\omega) \right| \leq C \left[1 + |\omega| \right]^{\log_M \sup_\omega Q(\omega) - K - \frac{1}{2}}. \qquad (27)$$

Therefore if $\sup_\omega Q(\omega) < M^{K-m-\frac{1}{2}}$, then $\psi_0(t)$ associated with a K-regular scaling filter is m times differentiable. However, since $Q(\omega)_{\omega=0} = \sqrt{M}$, $\psi_0(t)$ can be at most $(K-2)$ times differentiable. The sufficient condition for $\psi_0(t)$ to be m times differentiable is precisely given below [9]:

Fact 5 (Daubechies) *If $Q(z)$ is such that*

$$\sup_{\omega \in \mathbf{R}} \prod_{j=0}^{l} \left| Q\left(\frac{\omega}{M^j}\right) \right| < M^{l(K-m-\frac{1}{2})}, \tag{28}$$

then $\psi_0(t)$ is m times continuously differentiable.

The wavelets, being finite linear combination of translates of the scaling function, are as regular as the scaling function. In particular if $\sup_\omega Q(\omega) < M^{K-1}$, then the scaling function and wavelets are continuous.

Remark: Regularity as defined in this paper is a property of the scaling filter and *not* of the scaling function. It is easy to construct examples of of unitary scaling filters with different orders of regularity such that the scaling function corresponding to the less regular scaling filter is smoother than the scaling function corresponding to the more regular scaling filter[19].

4 Formula for Regular M-band Scaling Filters

We now describe the construction of K-regular M-band scaling filters *of minimal length.* We have seen that K-regularity is equivalent to $(M-1)(K-1)$ linear constraints on h_0, and that an arbitrary M-band scaling filter of length $N = MK$ is determined by $(M-1)(K-1)$ parameters. By imposing the regularity constraints on the general parametrization of unitary scaling filters, one expects to obtain K-regular scaling filters. However, there is no analytical method to solve the resultant set of $(M-1)(K-1)$ nonlinear equations (in the parameters) and until now numerical techniques have been the answer. Here we provide an explicit solution to the problem. We postulate the form of the scaling filter (Eqn. 25) and try to solve for the polynomial $Q(z)$ such that $H_0(z)$ is a unitary scaling filter. This approach is particularly simple because the unitariness conditions are linear in the autocorrelation of $Q(z)$.

With $N = MK$, $Q(z)$ is seen from Eqn. 25 to be a polynomial of degree $(K-1)$ in z^{-1}. With this choice $z^{K-1}Q(z)Q(z^{-1})$ is a polynomial of degree $(2K-2)$ in z. Let this polynomial be expanded in powers of $(z-1)^i$:

$$z^{K-1}Q(z)Q(z^{-1}) = \sum_{i=0}^{2K-2} p(i)(z-1)^i. \tag{29}$$

Therefore, if we can solve for the $p(i), i \in \{0, \ldots, 2K-2\}$ that give rise to a unitary scaling filter, and the resulting sequence is positive definite, $Q(z)$ can be obtained by spectral factorization. Thus solving for K-regular unitary scaling filters is equivalent to solving for the finite length positive definite sequence $p(i)$ under the unitariness constraint.

The unitariness condition $[\downarrow M]\,[H_0(z)H_0(z^{-1})] = 1$ is linear in $p(i)$. Since $H_0(z)$ is of length MK, this is equivalent to a set of K linear equations for the K unknowns $p(i)$. Fortunately, these linear equations always have a unique solution which is positive definite, thus giving rise to a K-regular h_0. In the following we give two *analytical approaches* to obtain the coefficients $p(i)$.

4.1 Explicit Formula based on Unitariness

This approach has the advantage that a clean formula for the $p(i)$ can be obtained. The essential idea is to take the autocorrelation on both sides of Eqn. 25 and write it in a convenient form, following which both sides are downsampled by M. By the unitariness assumption (Eqn. 10) the dependence of $p(i)$ on $H_0(z)H_0(z^{-1})$ disappears, giving a set of linear equations for $p(i)$ which can be analytically solved.

$$\frac{z^{MK}}{[z^M-1]^{2K}} H_0(z)H_0(z^{-1}) = \frac{1}{M^{2K}} \sum_{i=0}^{2K-2} p(i) \frac{z}{[z-1]^{2K-i}} \tag{30a}$$

$$\Rightarrow \quad \frac{z^K}{[z-1]^{2K}} = [\downarrow M] \left\{ \frac{1}{M^{2K}} \sum_{i=0}^{2K-2} p(i) \frac{z}{[z-1]^{2K-i}} \right\}. \tag{30b}$$

By expanding both sides as Taylor series about $z=\infty$, the coefficient of z^{-l} in the expansion of $\frac{z^K}{[z-1]^{2K}}$ must be equal to the coefficient of z^{-Ml} of the expansion of $\frac{1}{M^{2K}} \sum_{i=0}^{2K-2} p(i) \frac{z}{[z-1]^{2K-i}}$ for all $l \geq 0$. Since for $m<n$

$$\frac{z^m}{[z-1]^n} = z^{m-n} \frac{1}{[1-z^{-1}]^n} = \sum_{l=0}^{\infty} \binom{l+m-1}{n-1} z^{-l},$$

this implies that for all $l \geq 0$

$$\binom{l+K-1}{2K-1} = \frac{1}{M^{2K}} \sum_{i=0}^{2M-2} \binom{Ml}{2M-i-1} p(i).$$

The above equation can be explicitly solved for the $p(i)$'s (see [39]) to give

$$p(i) = (2K-1-i) \sum_{l=0}^{2K-2-i} \binom{2K-2-i}{l} (-1)^{i-l} B_{l+1} \quad , i \in \{0, \ldots, 2K-2\}, \tag{31}$$

where

$$B_l = \frac{M}{(2K-1)!} \prod_{k=l}^{K-1} (l^2 - k^2 M^2).$$

Example 1[Computation of $Q(z)Q(z^{-1})$]

<u>$K=2$</u> $B_l = \frac{M}{3!}(l^2 - M^2)$. In this case (from Eqn. 31) $p(0)=M$, $p(1)=M$ and $p(2)=$

$\frac{1}{6}M(1-M^2)$, and therefore

$$Q(z)Q(z^{-1}) = Mz^{-1}\left\{1+(z-1)+\frac{1}{6}(1-M^2)(z-1)^2\right\}$$
$$= \frac{M}{6}\left\{(1-M^2)z^{-1}+(4+2M^2)+(1-M^2)z\right\}$$

<u>$K=3$</u> $B_l = \frac{M}{5!}\cdot(l-M^2)(l^2-4M^2)$ and $p(0)=M$, $p(1)=2M$, $p(2)=\frac{1}{4}(5-M^2)$, $p(3)=\frac{1}{4}(1-M^2)$ and $p(4)=\frac{1}{5!}(1-5M^2+4M^4)$. Therefore $Q(z)Q(z^{-1}) = \sum_{k=-2}^{k=2} a_k z^k$ where

$$a_2 = a_{-2} = \frac{M}{5!}(1-5M^2+4M^4),$$
$$a_1 = a_{-1} = \frac{M}{5!}(26-10M^2-16M^4) \text{ and}$$
$$a_0 = \frac{M}{5!}(66+30M^2+24M^2).$$

For small M and N a similar formula for $p(i)$ may be obtained as follows. If we replace z by $ze^{\imath 2\pi k/M}$ in Eqn. 30, sum over $k \in \{0,\ldots,M-1\}$, and use Eqn. 9 we get

$$M\frac{z^{MK}}{\left[z^M-1\right]^{2K}} = \frac{1}{M^{2K}}\sum_{i=0}^{2K-2} p(i)\sum_{k=0}^{M-1}\frac{ze^{\imath 2\pi k/M}}{\left[ze^{\imath 2\pi k/M}-1\right]^{2K-i}} \quad (32a)$$

$$\Rightarrow \quad M^{2K+1}\frac{z^{MK-1}}{\left[z^M-1\right]^{2K}} = \sum_{i=0}^{2K-2} p(i)\sum_{k=0}^{M-1}\frac{e^{\imath 2\pi k/M}}{\left[ze^{\imath 2\pi k/M}-1\right]^{2K-i}} \quad (32b)$$

$$\Rightarrow \quad p(i) = \frac{M^{2K+1}}{i!}\frac{d^i}{dz^i}\left[\left\{\left[\frac{z-1}{z^M-1}\right]^{2K}\cdot z^{MK-1}\right\}\right]_{z=1} \quad (32c)$$

The last step follows by taking the residues on both sides at $z=1$. For arbitrary M and K, the first few values of $p(i)$ from the above formula are given below:

$$p(0)=M, \quad p(1)=M(K-1), \quad p(2)=\frac{1}{12}\left\{-M^3K+6K^2M-17KM+12M\right\}, \quad \text{etc.}$$

4.2 Formula based on Maximal Flatness Condition

One of the consequences of unitariness for any $H_0(z)$ satisfying Eqn. 25 is that $|H_0(\omega)|^2$ is flat of order $2K$ at zero frequency. Conversely, the *maximal flatness* property in conjunction with Eqn. 25 ensures that unitariness is satisfied. Therefore, by imposing maximal flatness of $H_0(z)$ in Eqn. 25 one obtains a formula for the $p(i)$'s which gives minimal length K-regular scaling filters. This technique has been used by Vaidyanathan to derive 2-band regular unitary scaling filters [44]. [1]

[1] One of the authors has also obtained a purely algebraic formulae for the impulse responses of the filters with regularity of order $K=2,3,4$ in [21] building upon work of Pollen [34]. These formulae yield *closed form* algebraic expressions for the filter taps; in the $K=4$ case this is a new result even for $M=2$.

It will be useful to move between two variables in the frequency representation of our filters: ω and $x = \text{Re}z = \cos\omega$. Zero frequency ($\omega = 0$) then corresponds to $x = 1$ with $x \in [-1, +1)$ for $\omega \in [0, 2\pi)$.

The M-band Haar scaling filter has impulse response $\frac{1}{\sqrt{M}}(1, 1, 1, \ldots, 1)$ so that its its magnitude squared frequency response is given by

$$\begin{aligned} E(\omega) = \left|\frac{1 + e^{-\imath\omega} + \ldots + e^{-\imath(M-1)\omega}}{M}\right|^2 &= \frac{1}{M^2}\left[M + \sum_{k=1}^{M-1} 2(M-k)\cos k\omega\right] \\ &= \frac{1}{M^2}\left[M + \sum_{k=1}^{M-1} 2(M-k)T_k(x)\right] \\ &\stackrel{\text{def}}{=} \mathcal{E}(x), \end{aligned} \tag{33}$$

where $T_k(x)$ is the k^{th} Chebyshev polynomial. For example, when $M = 3$, we have

$$E(\omega) = \left(\frac{1 + 2\cos\omega}{3}\right)^2 \quad \text{and} \quad \mathcal{E}(x) = \left(\frac{1 + 2x}{3}\right)^2.$$

Now let $\mathcal{P}(x) = Q(z)Q(z^{-1})$ and let $\mathcal{A}(x) = H_0(z)H_0(z^{-1})$. Then

$$\mathcal{A}(x) = \mathcal{E}^K(x)\mathcal{P}(x). \tag{34}$$

The flatness condition implies that

$$\mathcal{A}^{(i)}(x)_{x=1} = \begin{cases} M & i = 0 \\ 0 & i = 1, 2, \ldots, K-1\,; \end{cases} \tag{35}$$

We now need to determine $\mathcal{P}(x) = |Q(\omega)|^2$ from which we can obtain $Q(\omega)$. $\mathcal{P}(x)$ can be expanded in a Taylor series about $x = 1$. Now

$$\begin{aligned} \mathcal{P}^{(n)}(x)_{x=1} &= \left[\left(\frac{d}{dx}\right)^n \mathcal{A}(x)\,[\mathcal{E}(x)]^{-K}\right]_{x=1} \\ &= \sum_{i=0}^{n} \binom{n}{i} \left[\left(\frac{d}{dx}\right)^i \mathcal{A}(x)\right]_{x=1} \left[\left(\frac{d}{dx}\right)^{n-i} [\mathcal{E}(x)]^{-K}\right]_{x=1} \\ &= M\left[\left(\frac{d}{dx}\right)^n [\mathcal{E}(x)]^{-K}\right]_{x=1}, \end{aligned} \tag{36}$$

and therefore

$$\mathcal{P}(x) = M\sum_{n=0}^{K-1} \left\{\frac{1}{n!}\left(\frac{d}{dx}\right)^n [\mathcal{E}(x)]^{-K}\right\}_{x=1} (x-1)^n. \tag{37}$$

Example 2[K-regular 3-band Case] In this case for K-regularity

$$\mathcal{E}(x) = \left(\frac{1+2x}{3}\right)^2 \quad \text{and} \quad \mathcal{A}(x) = \left(\frac{1+2x}{3}\right)^{2K} \mathcal{P}(x).$$

From Eqn. 37 it can be shown that (see [22])

$$\mathcal{P}^{(n)}(x)_{x=1} = \left[\left(\frac{d}{dx}\right)^n [\mathcal{E}(x)]^{-K}\right]_{x=1} = (-1)^n \, \frac{(2K+n-1)!}{(2K-1)!} \, \left(\frac{2}{3}\right)^n.$$

$$\Rightarrow \quad \mathcal{P}(x) = 3\sum_{n=0}^{K-1} \binom{2K+n-1}{2K-1} \left(\frac{-2}{3}\right)^n (x-1)^n.$$

For arbitrary M, and small values of K one can evaluate $\mathcal{P}(x)$ explicitly. For example when $K=2, \quad \mathcal{P}(x) = M + \frac{1}{3}M(1-M^2)(x-1)$.

4.3 Spectral Factorization

If $z^{-(K-1)} \sum_{i=0}^{K-1} p(i)(z-1)^i$ has for coefficients (of powers of z) a positive definite sequence, then $Q(z)$ can be obtained by spectral factorization. Positive definiteness of this sequence can be inferred from the general Lagrange interpolation arguments in [40]. There is a degree of freedom in the choice of $Q(z)$ depending on which spectral factors are chosen. One may choose a minimum phase, maximum phase, or mixed solution. For each such choice of $Q(z)$ one has a corresponding K-regular unitary scaling filter. The minimal phase solutions for K-regular unitary scaling filters for $M=3$ and $M=4$ are given in Table 1. For $K=2$, the minimal phase and maximal phase solutions (there are only two solutions in this case) for arbitrary M is given by the following formula:

$$H_0(z) = \left[\frac{1+z^{-1}+\ldots+z^{-(M-1)}}{M}\right]^2 (q(0)+q(1)z^{-1}),$$

where

$$q(0) = \frac{\sqrt{M}}{2}\left[1 \pm \sqrt{\frac{2M^2+1}{3}}\right] \quad \text{and} \quad q(1) = \frac{\sqrt{M}}{2}\left[1 \mp \sqrt{\frac{2M^2+1}{3}}\right].$$

Figs. 4-6 show the scaling functions, their Fourier transform, and the Fourier transform of the scaling filter, for 3-band, 4-band and 5-band case for $K = 2, 3, 4$ and 5. Notice that the shape of an M-band K-regular scaling function is largely determined by its regularity, K. Notice also that the Fourier transforms of the scaling functions vanish at multiples of 2π, and that $H_0(\omega)$ does not vanish for $|\omega| < \frac{\pi}{M}$ (in fact the first zero is at $\frac{2\pi}{M}$). This fact implies (see Section 6) that all K-regular minimal length scaling filters give rise to orthonormal wavelet bases.

5 Construction of Regular Wavelet Filters

Given a K-regular unitary scaling filter there seems to be no systematic procedure to generate *all* possible wavelet filters with the same length. In the 2-band case, the wavelet filter is uniquely

Table 1: K-Regular M-band Minimal Length Unitary Scaling Filters

$M=3$	n	$h_0(n)$	$M=3$	n	$h_0(n)$	$M=3$	n	$h_0(n)$
$K=2$	0	0.33838609728386	$K=3$	0	0.20313514584456	$K=6$	0	0.04641991275121
	1	0.53083618701374		1	0.42315033910807		1	0.16394657299264
	2	0.72328627674361		2	0.70731556228155		2	0.40667150052122
	3	0.23896417190576		3	0.44622537783130		3	0.56561987503637
	4	0.04651408217589		4	0.19864508103414		4	0.58223034773984
	5	-0.14593600755399		5	-0.17723527558292		5	0.24390438994869
$K=5$	0	0.07550761756143		6	-0.07201025448623		6	-0.03360979671399
	1	0.23086070821719		7	-0.04444515095259		7	-0.25350741685252
	2	0.51304535032014		8	0.04726998249100		8	-0.08274027041541
	3	0.59269796491023	$K=4$	0	0.12340698195349		9	-0.00156787261030
	4	0.50343156427108		1	0.31789563892953		10	0.11605073148585
	5	0.07274582768779		2	0.62131686335095		11	0.00346097586136
	6	-0.11559776131042		3	0.56142607070711		12	0.00040170813801
	7	-0.21804646388388		4	0.36890783202512		13	-0.03676774192987
	8	0.00692356260197		5	-0.08625807908307		14	0.00823961325941
	9	0.02913316570545		6	-0.12777980080646		15	0.00008644258833
	10	0.07286749987661		7	-0.13375920464072		16	0.00539777575368
	11	-0.02130382202714		8	0.05875903404127		17	-0.00218593998563
	12	-0.00439071767705		9	0.02029701733548			
	13	-0.01176303929137		10	0.02430600287569			
	14	0.00593935060686		11	-0.01646754911953			

$M=4$	n	$h_0(n)$	$M=4$	n	$h_0(n)$	$M=4$	n	$h_0(n)$
$K=2$	0	0.26978904939721	$K=4$	0	0.08571412050958	$K=5$	0	0.04916991424487
	1	0.39478904939721		1	0.19313899295294		1	0.12913015554835
	2	0.51978904939721		2	0.34917971394336		2	0.26140970524347
	3	0.64478904939721		3	0.56164878348085		3	0.46212341604513
	4	0.23021095060279		4	0.49550221952707		4	0.50348969444395
	5	0.10521095060279		5	0.41456599638527		5	0.49742757908607
	6	-0.01978904939721		6	0.21903222760227		6	0.35826639102137
	7	-0.14478904939721		7	-0.11453658682193		7	0.02935921939015
$K=3$	0	0.15083145463571		8	-0.09529322382982		8	-0.06205420421862
	1	0.28192600003506		9	-0.13069539487629		9	-0.17204166252712
	2	0.44427054543441		10	-0.08275002028156		10	-0.16539775306492
	3	0.63786509083375		11	0.07198039995437		11	0.03112914045751
	4	0.41021527232597		12	0.01407688379317		12	0.01081024188105
	5	0.27302618152727		13	0.02299040553808		13	0.05413053935774
	6	0.07333709072858		14	0.01453807873593		14	0.05420808584699
	7	-0.18885200007012		15	-0.01909259661330		15	-0.02997902951088
	8	-0.06104672696168					16	-0.00141564635125
	9	-0.05495218156233					17	-0.00864661146505
	10	-0.01760763616298					18	-0.00848642904691
	11	0.05098690923636					19	0.00736725361809

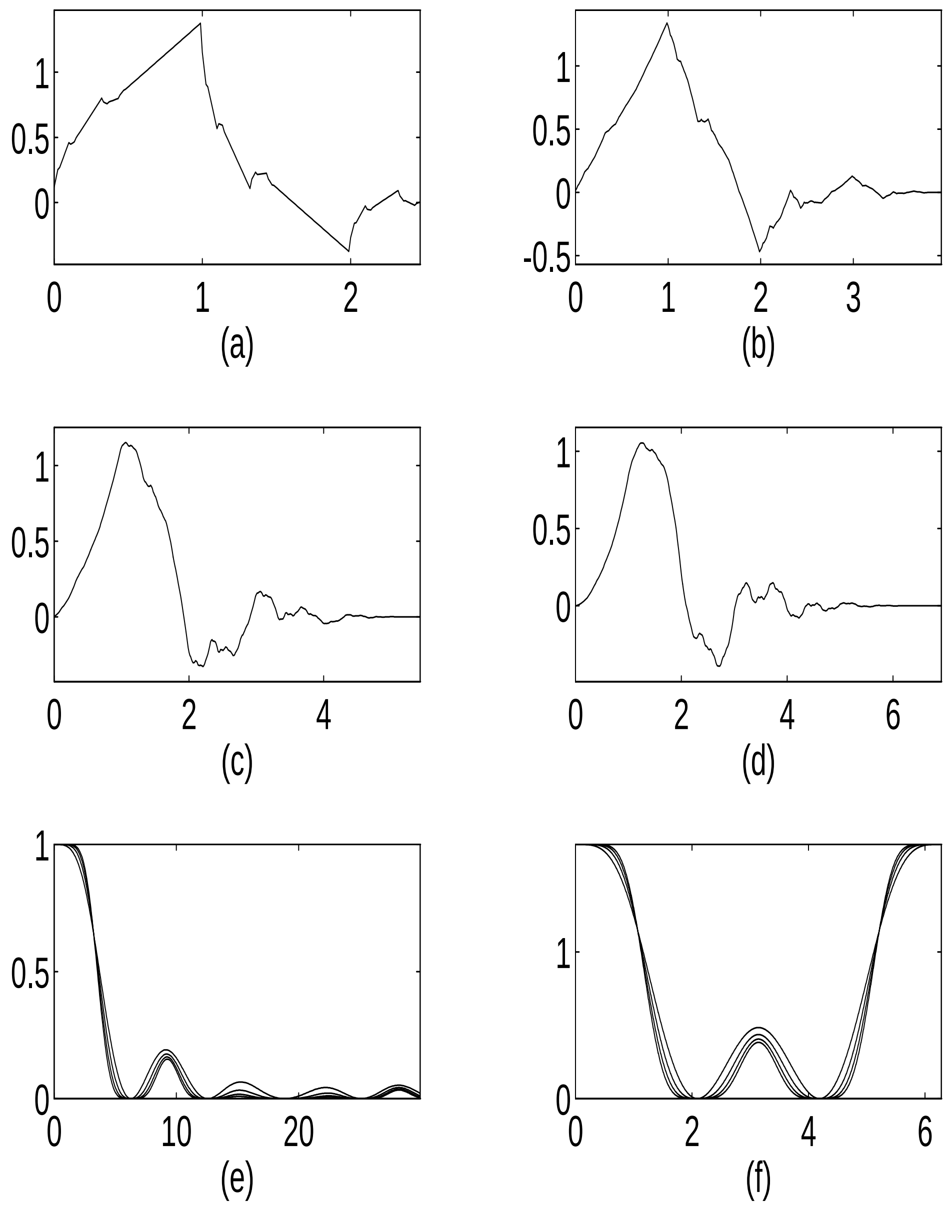

Figure 4: K-regular 3-band Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

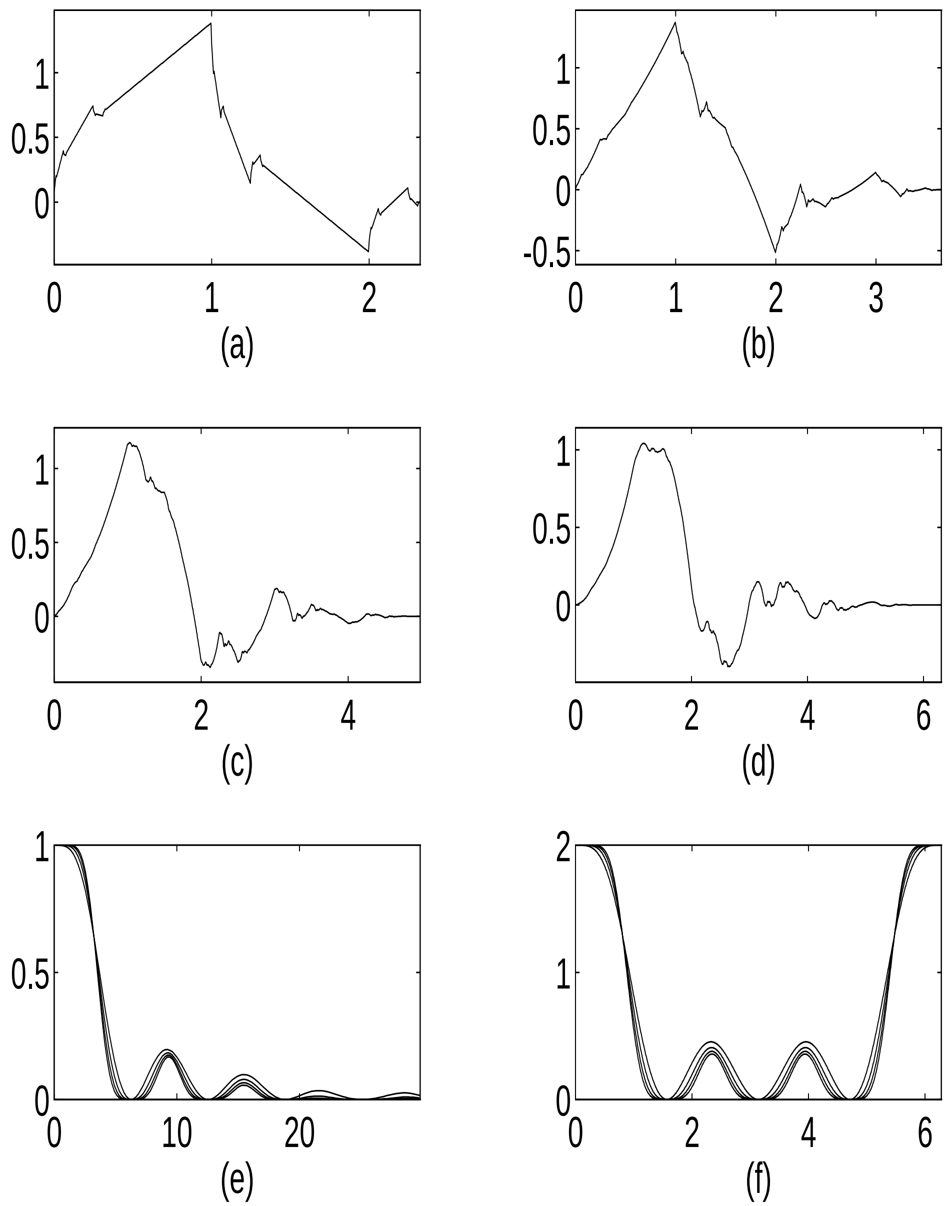

Figure 5: K-regular 4-band Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

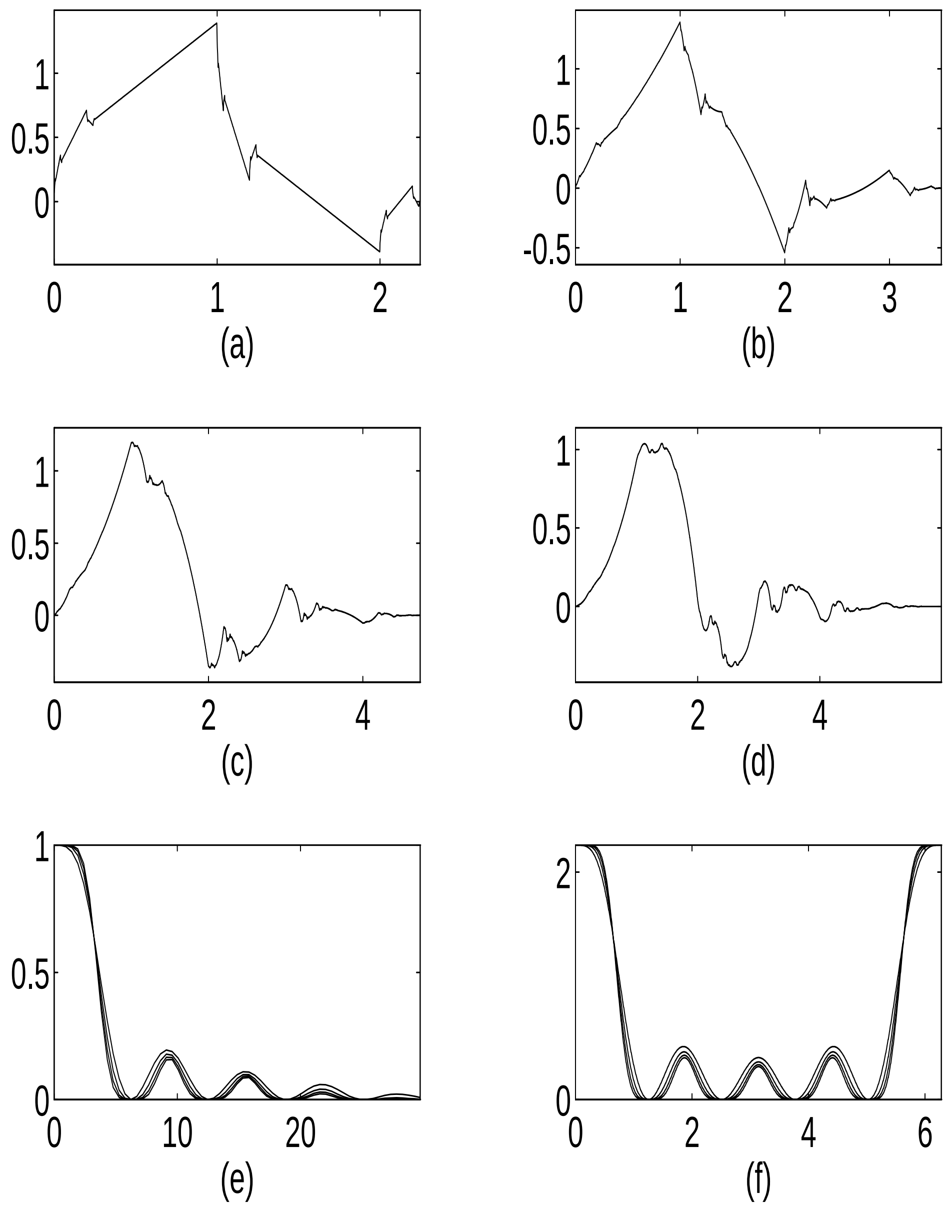

Figure 6: K-regular 5-band Scaling Functions: (a) $\psi_0(t)$ for $K = 2$ (b) $\psi_0(t)$ for $K = 3$ (c) $\psi_0(t)$ for $K = 4$ (d) $\psi_0(t)$ for $K = 5$ (e) $\hat{\psi}_0(\omega)$ for $K = 2$ through $K = 5$ with maximal flatness of the Fourier transform increasing with K (f) $H_0(\omega)$ for $K = 2$ through $K = 5$

given (modulo translation) by the scaling filter by the following well-known formula: $h_1(n) = (-1)^n h_0(N-1-n)$, where N is the length of the scaling filter. However, in the M-band case there is a certain degree of freedom in the choice of the wavelet filters. As described in Section 2 all wavelet filters such that the McMillan degree of $\mathbf{H}(z)$ is equal to the McMillan degree of the unitary scaling filter can be obtained from Eqn. 12 by choosing V_0. This method for constructing wavelet filters and wavelets has been suggested in [17, 52, 23]. The only restriction on V_0 is that it be orthogonal and that the first row is $\left[\begin{array}{ccc} 1/\sqrt{M} & \ldots & 1/\sqrt{M} \end{array}\right]$. Therefore, two potential choices for V_0 are the Type 2 Discrete-Cosine-Transform (DCT) matrix [35] and the DFT matrix. With the DFT matrix as V_0, the wavelet filters have complex-valued coefficients and therefore the wavelet are also complex-valued. Since there is a degree of freedom in the choice of wavelet filters, one expects to design wavelet filters suited for a given application. In the following a state-space characterization of wavelet tight frames is given and a formula for wavelet filters that depends on an arbitrary orthogonal matrix Θ is given.

5.1 State-Space Approach to M-band Wavelets

Given any FIR unitary $\mathbf{H}(z)$ the well-known Kalman-Yakubovich Lemma of linear-systems theory (for a treatment relevant to our discussion see [10]) ensures the existence of matrices A, B, C and D (of sizes $L \times L$, $L \times M$, $M \times L$ and $M \times M$ respectively) such that

$$\mathbf{H}(z) = C(zI - A)^{-1}B + D \qquad \text{and} \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]^T \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = I \tag{38}$$

The matrices A,B, C and D constitute the state-space description of $\mathbf{H}(z)$ in a special basis where $\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ is unitary. Using this result we obtain a new characterization of wavelet tight frames.

Theorem 2 *Let Y be a constant, $(L+M) \times (L+M)$ unitary matrix with an $L \times L$ nilpotent submatrix (i.e., a matrix with all eigenvalues zero). Without loss of generality (by permutation) assume Y is of the form*

$$Y = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \tag{39}$$

where A is nilpotent (i.e., $A^i = 0$ for some i). Furthermore, for some row $\left[\begin{array}{cc} c & d \end{array}\right]$ of $\left[\begin{array}{cc} C & D \end{array}\right]$ let

$$c(I-A)^{-1}B + d = \left[\begin{array}{ccc} \frac{1}{\sqrt{M}} & \cdots & \frac{1}{\sqrt{M}} \end{array}\right] \tag{40}$$

Then there exists a compactly supported wavelet tight frame with $\psi_i(t)$ *supported in* $\left[0, \frac{N-1}{M-1}\right]$*, such that* $[A, B, c, d]$ *is a realization of the polyphase representation of the scaling filter*

$$H_0(z) = c(zI - A)^{-1}B + d \tag{41}$$

Conversely, given an arbitrary wavelet tight frame with support in $\left[0, \frac{N-1}{M-1}\right]$*, there exists a unitary* Y *with the above properties.*

We now describe the construction of K-regular wavelet filters using a state-space approach. The basic idea is to construct the state-space wavelet matrix. Notice that the scaling filter determines part of the state space description matrix. Let $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ be a minimal realization of the polyphase representation of the scaling filter, i.e., of $\hat{\mathbf{H}}(z) = \left[\begin{array}{cccc} H_{0,0}(z) & H_{0,1}(z) & \ldots & H_{0,M-1}(z) \end{array}\right]$. Now for any choice of wavelet filters such that $\mathbf{H}(z)$ has the same McMillan degree $L - 1 = K - 1$, there exists a unitary state-space wavelet matrix Y which may be partitioned as $\left[\begin{array}{c} Y_1 \\ \hline Y_2 \end{array}\right] = \left[\begin{array}{cc} A & B \\ C_1 & D_1 \\ \hline C_2 & D_2 \end{array}\right]$. The polyphase component vector of $H_0(z)$ the scaling filter is given by

$$C_1(zI - A)^{-1}B + D_1 = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{42}$$

and therefore there exists an invertible transformation T such that

$$\left[\begin{array}{cc} A & B \\ C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc} T\hat{A}T^{-1} & T\hat{B} \\ \hat{C}T^{-1} & \hat{D} \end{array}\right]. \tag{43}$$

The transformation T will be called the *balancing* transformation (because the particular state-space realization is known as a balanced realization [32, 33]). Given any realization of the scaling filter and its balancing transformation we can obtain the state-space wavelet matrix for all but its last $(M - 1)$ rows.

The balancing transform can be obtained from the controllability matrix of *any* minimal realization of $\hat{H}(z)$. The controllability matrix for any state-space realization of the system $\hat{H}(z)$ is defined

$$W_c = \sum_k \hat{A}^k \hat{B} \hat{B}^T (\hat{A}^k)^T. \tag{44}$$

W_c is a symmetric matrix and can be obtained by solving the the Lyapunov equations ([25])

$$W_c = \hat{A} W_c \hat{A}^T + \hat{B}\hat{B}^T. \tag{45}$$

Moreover if $\hat{H}(z)$ is unitary on the unit circle then W_c also satisfies the following equations ([25])

$$\hat{C}W_c\hat{C}^T + \hat{D}\hat{D}^T = I \tag{46a}$$

$$\hat{C}W_cA^T + \hat{D}\hat{B}^T = 0 \tag{46b}$$

By the unitariness of Y, $Y_1Y_1^T = I$ and T must be chosen such that this equation is satisfied. This implies that

$$AA^T + BB^T = I; \quad AC_1^T + BD_1^T = 0; \quad C_1C_1^T + D_1D_1^T = 1. \tag{47}$$

Theorem 3 *Let $[\hat{A}, \hat{B}, \hat{C}, \hat{D}]$ be a minimal realization of $\hat{H}(z)$. Let $\hat{W}_c$ be the controllability matrix of $\hat{H}(z)$. Then $T = W_c^{-\frac{1}{2}}$ is the balancing transformation for $\hat{H}(z)$.*

Proof: Eqn. 47 follows from Eqn. 43, Eqn. 45 and Eqn. 46:

$$\begin{aligned} AA^T + BB^T &= T\hat{A}T^{-1}(T^{-1})^T\hat{A}^TT + T\hat{B}\hat{B}^TT^T \\ &= W_c^{-\frac{1}{2}}\hat{A}W_c\hat{A}^TW_c^{-\frac{1}{2}} + W_c^{-\frac{1}{2}}\hat{B}\hat{B}^TW_c^{-\frac{1}{2}} \\ &= W_c^{-\frac{1}{2}}\left[W_c - \hat{B}\hat{B}^T\right]W_c^{-\frac{1}{2}} + W_c^{-\frac{1}{2}}\hat{B}\hat{B}^TW_c^{-\frac{1}{2}} \\ &= I \end{aligned} \tag{48}$$

$$\begin{aligned} AC_1^T + BD_1^T &= W_c^{-\frac{1}{2}}\hat{A}(W_c^{-\frac{1}{2}})^{-1}(W_c^{-\frac{1}{2}})^{-1}\hat{C}^T + W_c^{-\frac{1}{2}}\hat{B}\hat{D}^T \\ &= W_c^{-\frac{1}{2}}\left[\hat{A}W_c\hat{C}^T + \hat{B}\hat{D}^T\right] = 0 \end{aligned} \tag{49}$$

$$\begin{aligned} C_1C_1^T + D_1D_1^T &= \hat{C}(W_c^{-\frac{1}{2}})^{-1}(\hat{C}(W_c^{-\frac{1}{2}})^{-1})^T + \hat{D}\hat{D}^T \\ &= \hat{C}W_c\hat{C}^T + \hat{D}\hat{D}^T = I. \end{aligned} \tag{50}$$

□

It is interesting to note that though in principle T could have depended on any of the four state-space matrices that describe the scaling filter, it really depended only upon the state and input matrices (A and B respectively).

5.1.1 SVD completion of Y_1

We now describe *one way* to construct the wavelet filters in this state-space setting by using the singular-value-decomposition (SVD). Given right unitary Y_1, one has to find Y_2 such that

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{51}$$

Define the symmetric positive definite matrix $X = Y_2 Y_2^T$ so that $X = I - Y_1 Y_1^T$. Let $X = U \Sigma U^T$ be the SVD of X. Σ is a diagonal matrix of positive entries. Then a general solution for Y_2 is given by $Y_2 = U(\Sigma)^{\frac{1}{2}}\Theta$ where Θ is an arbitrary $M-1 \times M-1$ constant unitary matrix. This follows since $Y_2 Y_2^T = U(\Sigma)^{\frac{1}{2}}\Theta\Theta^T(\Sigma)^{\frac{1}{2}}U^T = X$. The number of degrees of freedom in the choice of the wavelet filters is $\binom{M-1}{2}$.

6 Necessary and Sufficient Conditions For Orthonormality

When is a WTF an orthonormal basis? Stated differently, what are the conditions on the scaling filter such that the WTF constructed from it (as in Section 2) forms an ON basis? It is relatively easy to see that *if* the scaling function and its integer translates form an orthonormal system, then the WTF is an ON basis. First notice that

$$\begin{aligned}\int_{\mathbf{R}} \psi_i(t)\psi_j(t-k)\,dt &= \sum_{m,n} h_i(m)h_j(n)\left[M\int_{\mathbf{R}} \psi_0(Mt-m)\psi_0(Mt-n-Mk)\,dt\right] \\ &= \sum_n h_i(n+Mk)h_j(n) \\ &= \delta(i-j)\delta(k). \end{aligned} \tag{52}$$

and therefore $W_{i,0} \perp W_{j,0}$, for $i \neq j$. Moreover $W_{i,0}$ is equipped with ON basis. For any J one readily sees that $W_{i,J} \perp W_{j,J}$, for $i \neq j$ and hence the result follows.

When $M = 2$, Cohen [5] and Lawton [27] have independently obtained characterizations of the scaling filter such that the scaling function and its translates form an orthonormal system. We now extend these results to the M-band case. Let

$$a(n) = \int_{\mathbf{R}} \psi_0(t)\psi_0(t-n)\,dt. \tag{53}$$

By taking Fourier transforms on both sides, $a(n) = \delta(n)$ iff

$$\sum_k \left|\hat{\psi}_0(\omega + 2\pi k)\right|^2 = 1. \tag{54}$$

Definition 4 *A compact set* Γ *is congruent to* $[-\pi,\pi](\mathrm{mod} 2\pi)$*, if the measure of* Γ *is* 2π *and for every point* $\omega \in [-\pi,\pi]$*, there exists an* $n \in \mathbf{Z}$ *such that* $\omega + 2\pi n \in \Gamma$.

Theorem 4 *The following conditions are equivalent:*

1. $\psi_0(t)$ *and its translates are orthonormal (i.e* $a(n) = \delta(n)$*).*

2. There exists Γ congruent to $[-\pi, \pi]$, containing a neighborhood of zero such for $\omega \in \Gamma$,

$$\hat{\psi}_0(\omega) \geq C > 0 \tag{55}$$

3. There exists Γ (as in 2) such that for $\omega \in \Gamma$,

$$\inf_{j>0} \inf_{\omega \in \Gamma} \left| \frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^j}\right) \right| = B > 0. \tag{56}$$

4. $a(n) = \delta(n)$ is the unique solution of the equation

$$a(k) = \sum_n a(Mk+n) \left[\sum_m h_0(m) h_0(n+m) \right]. \tag{57}$$

5. $A(\omega) = 1$ is the unique solution of the equation

$$A(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} \left| H_i\left(\frac{\omega + 2\pi k}{M}\right) \right|^2 A\left(\frac{\omega + 2\pi k}{M}\right). \tag{58}$$

6. There is no non-trivial cycle Π of the map $\omega \mapsto M\omega (\text{mod } 2\pi)$, such that $H_0(\omega) = 1$, for all $\omega \in \Pi$.

Proof: 1 implies 2 by exactly the same arguments as the 2-band case in [8, p. 182]. 2 implies 1 follows by using the arguments in [8, p. 184] in conjunction with Eqn. 9. 4 and 5 are equivalent via the Fourier transform. 2 implies 3 because for any j, and $\omega \in \Gamma$, from Eqn. 55 (since $\left|\frac{H_0(\omega)}{\sqrt{M}}\right| \leq 1$ and $\hat{\psi}_0(\omega) \leq 1$)

$$\left| \frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^j}\right) \right| \geq \prod_{i=1}^{j} \left| \frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^i}\right) \right| = \frac{\hat{\psi}_0(\omega)}{\hat{\psi}_0(\frac{\omega}{M^j})} \geq C > 0.$$

3 implies 2 and can be seen as follows:

$$\left| H_0(\omega) - \sqrt{M} \right| = \left| H_0(\omega) - H_0(0) \right| \leq \sum_{n=0}^{N-1} |h_0(n)| \left| e^{-\imath\omega n} - 1 \right| \leq A |\omega|$$

for some $A > 0$. Hence for $\omega \in \Gamma$ and k sufficiently large, $\left| \frac{1}{\sqrt{M}} H_0\left(\frac{\omega}{M^k}\right) \right| \geq 1 - A \left| \frac{\omega}{M^k} \right| \geq e^{-A\left|\frac{\omega}{M^j}\right|}$ and therefore

$$\begin{aligned} \left| \hat{\psi}_0(\omega) \right| &\geq B^{k-1} \prod_{j=k}^{\infty} \left[1 - A \left| \frac{\omega}{M^j} \right| \right] \\ &\geq B^{k-1} e^{-A \sum_{j=k}^{\infty} \left| \frac{\omega}{M^j} \right|} \geq C. \end{aligned} \tag{59}$$

4 implies 1 since from Eqn. 17 $a(n) = \int_{\mathbf{R}} \psi_0(t)\psi_0(t+n)\,dt = \sum_n a(Mk+n)\left[\sum_m h_0(m)h_0(n+m)\right]$, and by hypothesis $a(n) = \delta(n)$ is the only solution.

One proves that 1 implies 5 by contradiction. Let $\{\psi_0(t-k)\}$ be an ON system and let there exist $A(\omega) \neq 1$ that satisfies Eqn. 58. We may assume $A(\omega) > 0$ by adding an appropriate constant to it if necessary (Eqn. 58 will still be satisfied). Define

$$H_0'(\omega) = H_0(\omega)\sqrt{\frac{A(\omega)}{A(M\omega)}}.$$

Then $H_0'(\omega)$ is also a unitary scaling filter since from Eqn. 58

$$\frac{1}{M}\sum_{k=0}^{M-1}\left|H_0'\left(\frac{\omega+2\pi k}{M}\right)\right|^2 = 1.$$

Let $\psi_0'(t)$ be the corresponding scaling function (possibly infinitely supported).

$$\hat{\psi}_0'(\omega) = \prod_1^{\infty}\left[\frac{1}{\sqrt{M}}H_0'\left(\frac{\omega}{M^J}\right)\right] = \hat{\psi}_0(\omega)/\sqrt{A(\omega)}.$$

Since the zero sets of H_0 and H_0' coincide ($A(\omega) > 0$), if $\hat{\psi}_0(\omega)$ is bounded below on a compact set Γ then so is $\hat{\psi}_0{}'(\omega)$. Therefore, $\{\psi_0{}'(t-k)\}$ is also an orthonormal system and

$$1 = \sum_k \left|\hat{\psi}_0'(\omega+2\pi k)\right|^2 = \sum_k \left|\hat{\psi}_0(\omega+2\pi k)\right|^2/A(\omega) = 1. \tag{60}$$

Therefore $A(\omega) = 1$ (recall $A(\omega)$ it is periodic), a contradiction. Equivalence of 5 and 6 can be proved based on ideas for the 2-band case originally developed by Cohen in his PhD thesis [4]. For a simple argument in the M-band case see [14]). □

The characterizations of orthonormality may be used to show that a particular wavelet basis constructed is orthonormal. Of all the characterizations of orthonormality, Eqn. 57 is the easiest to verify. It says that given an unitary scaling filter, the corresponding wavelet basis is ON iff the *Lawton matrix* (after Lawton who constructed it for the 2-band case [27]) defined below has a unique eigenvector of eigenvalue 1. If $r(n)$ is the autocorrelation sequence of $h_0(n)$ (of length $N = MK$), Eqn. 57 becomes

$$a(n) = \sum_l r(l)a(Mn-l) \tag{61}$$

The Lawton matrix Q is defined by

$$q_{i,j} = \begin{cases} r(M(i-1)) \text{ for } j = 1 \\ r(M(i-1)+j-1) + r(M(i-1)-j+1) \text{ for } 2 \leq j \leq N \end{cases}$$

If $v = \left[\begin{array}{ccc} a(0) & a(1) \ldots & a(N-2) \end{array} \right]^T$, Eqn. 61 becomes $Qv = v$. In general the Q matrix will be of the form

$$\left[\begin{array}{cccc} 1 & r(1)+r(-1) & \ldots & r(N-2)+r(-N+2) \\ 0 & r(M+1)+r(M-1) & \ldots & r(M+N-2)+r(M-N+2) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & r(M(N-2)+1)+r(M(N-2)-1) & \ldots & r((M+1)(N-2))+r((M-1)(N-2)) \end{array} \right].$$

$v = [1, 0, \ldots, 0]^T$ is always an eigenvector of Q with eigenvalue 1. If there exists any other eigenvector for Q, then $\{\psi_0(t-k)\}$ do not give rise to an orthonormal system (and the WTF may not be an ON basis).

There is a well-known sufficient condition for orthonormality in the 2-band case due to Mallat [29] which is easy to verify and stated in terms of $H_0(\omega)$. This condition can be generalized immediately to the M-band case and we have the following corollary of Theorem 4, a proof of which may be found in [14]. The essential idea is that in this case one can show $\hat{\psi}_0(\omega)$ does not vanish on the compact set $\Gamma = [-\pi, \pi]$.

Corollary 1 *If $H_0(\omega)$ does not vanish for $|\omega| \leq \frac{\pi}{M}$, then the wavelet basis generated from it is orthonormal.*

Remark: For the K-regular, minimal length M-band case, $\{\psi_0(t-k)\}$ is always an orthonormal system and hence the wavelet bases constructed in this paper are always orthonormal. Indeed in Figs. 4-6 it is clear that $H_0(\omega)$ has no zero for $|\omega| \leq \frac{\pi}{M}$ and orthonormality of the wavelet basis is immediately inferred.

7 Conclusion

This paper generalizes the minimal length K-regular 2-band wavelets of Daubechies to the M-band case. Several equivalent characterizations of K-regularity are given and their significance explained. Using two different approaches, an explicit formula for the magnitude-squared response of the unitary scaling filter that gives rise to minimal length K-regular M-band wavelets are obtained. By spectral factorization we obtain formulas for the scaling filter coefficients themselves. Wavelet bases are characterized using state-spaces techniques and a state-space characterization of wavelet filters associated with any given scaling filter is obtained. Both the state-space approach and the factorization based approach [17, 52, 23] may be used to design regular M-band wavelet filters. Wavelet bases constructed from unitary scaling filters are in general tight frames. Necessary and sufficient conditions for an M-band wavelet tight frame to be orthonormal are also given. All the minimal length, K-regular, M-band wavelet bases constructed are orthonormal.

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References

[1] V. Belevitch. *Classical Network Theory.* Holden-Day Inc., San Francisco, CA, 1968.

[2] M. Bellanger, B. Bonnerot, and M. Coudreuse. Digital Filtering Using Polyphase Network: Application to Sample Rate Alteration and Filter Banks. *IEEE Trans. on ASSP*, pages 109–114, April 1976.

[3] C. K. Chui. *An Introduction to Wavelets.* Academic Press, 1992.

[4] A. Cohen. *Ondelettes, analyses Multiresolution, et Traitement Numerique du Signal.* PhD thesis, 1990.

[5] A. Cohen, I.Daubechies, and J.C. Feauveau. Biorthogonal Bases of Compactly Supported Wavelets. *Comm. in Pure and Applied Math.*, 1992.

[6] I. Daubechies. Orthonormal Bases of Compactly Supported Wavelets. *Communications in Pure and Applied Mathematics*, (41):909–996, 1988.

[7] I. Daubechies. The Wavelet Transform, Time-Frequency Localization and Signal Analysis. *IEEE Trans. on Information Theory*, 36:961–1005, 1990.

[8] I. Daubechies. *Ten Lectures on Wavelets.* SIAM, 1992. CBMS Lecture Series.

[9] I. Daubechies and J. C. Lagarias. Two-Scale Difference Equations i : Existence and Global Regularity of Solutions. *SIAM Journal of Math. Anal.*, 22:1388–1410, 1991.

[10] Z. Doganata, P. P. Vaidyanathan, and T. Q. Nguyen. General Synthesis Procedures for FIR Lossless Transfer Matrices, for Perfect-Reconstruction Multirate Filter Bank Applications. *IEEE Trans. on ASSP*, 36(10):1561–1574, 1988.

[11] R. J. Duffin and R. C. Schaeffer. A Class of Non-Harmonic Fourier Series. *Trans. of American Math. Society*, pages 341–366, 1952.

[12] A. Fettweis, J. A. Nossek, and K. Meerkötter. Reconstruction of signals after filtering and sampling rate reduction. In *Proceedings of ICASSP '84*, volume 1, pages 11.7.1–11.7.4.

[13] R. A. Gopinath. Wavelet Transforms and Time-Scale Analysis of Signals. Master's thesis, Rice University, Houston, TX-77251, 1990.

[14] R. A. Gopinath. *Wavelet and Filter Banks - New Results and Applications*. PhD thesis, Rice University, Houston, TX-77251, August 1992.

[15] R. A. Gopinath and C. S. Burrus. On Upsampling, Downsampling and Rational Sampling Rate Filter Banks. Technical Report CML TR91-25, Computational Mathematics Laboratory, Rice University, 1991. IEEE Trans. on SP, to appear April 1994.

[16] R. A. Gopinath and C. S. Burrus. State-space Approach to Multiplicity M Orthonormal Wavelet Bases. Technical Report CML TR91-22, Computational Mathematics Laboratory, Rice University, 1991. Submitted to IEEE Trans. on SP.

[17] R. A. Gopinath and C. S. Burrus. Wavelets and Filter Banks. Technical Report CML TR91-20, Computational Mathematics Laboratory, Rice University, September 1991. Appears in "Wavelets - A Tutorial in Theory and Applications", edited by C.K.Chui, published by Academic Press.

[18] R. A. Gopinath and C. S. Burrus. Factorization Approach to Unitary Time-Varying Filter Banks. Technical Report CML-TR92-23, Rice University, December 1992. Submitted to IEEE Trans. on SP.

[19] R. A. Gopinath and C. S. Burrus. On Cosine-Modulated Wavelet Orthonormal Bases. Technical Report CML TR92-06, Computational Mathematics Laboratory, Rice University, Mar 1992. IEEE Trans. in Image Processing, to appear.

[20] P. Goupillaud, A. Grossman, and J. Morlet. Cyclo-octave and Related Transforms in Seismic Signal Analysis. *Geoexploration*, (23), 1984.

[21] P. Heller. Higher Rank Daubechies' Wavelets. Technical Report AD911204, Aware Inc., Cambridge, MA 02142, 1991.

[22] P. Heller. Regular M-band Wavelets. Technical Report AD920608, Aware Inc., Cambridge, MA 02142, 1992.

[23] P. Heller, H. W. Resnikoff, and R. O. Wells. Wavelet Matrices and the Representation of Discrete Functions. In C. K. Chui, editor, *Wavelets: A Tutorial in Theory and Applications*, pages 15–51. Academic Press, 1992.

[24] I.Daubechies and J. C. Lagarias. Two-Scale Difference Equations ii : Local Regularity, Infinite Products of Matrices and Fractals. *SIAM Journal of Math. Anal.*, 24, 1991.

[25] T. Kailath. *Linear Systems.* Prentice Hall, Englewood Cliffs, New Jersey, 1980.

[26] A. Latto and E. Tenenbaum. Les Ondelettes à Support Compact et La Solution Numerique de l'equation Burgers. *C.R. Acad. of Sci., Paris, t. 311, Serie I*, pages 903–909, 1990.

[27] W. M. Lawton. Necessary and Sufficient Conditions for Existence of Orthornomal Wavelet Bases. *Journal of Math. Physics*, 1990.

[28] W. M. Lawton. Tight Frames of Compactly Supported Wavelets. *Journal of Mathematical Physics*, 1990.

[29] S. Mallat. Multiresolution approximation and wavelets. *Trans. of American Math. Soc.*, (315):69–88, 1989.

[30] S. Mallat. A Theory for Multiscale Signal Decomposition : The Wavelet Representation. *IEEE Trans. on Pattern and Machine Intelligence*, 11(7):674–693, 1989.

[31] H. Malvar. *Signal Processing with Lapped Transforms.* Artech House, 1992.

[32] C. T. Mullis and R. A. Roberts. Synthesis of Minimum Roundoff Fixed Point Digital Filters. *IEEE Trans. on CAS*, pages 551–562, 1976.

[33] L. Pernebo and L.M. Silverman. Balanced Realization of Multivariate Linear Systems. *IEEE Trans. on Automatic Control*, 27:382, 1982.

[34] D. Pollen. Linear one-dimensional scaling functions. Technical Report AD900101, Aware Inc., Cambridge, MA, 1990.

[35] K. R. Rao and P. Yip. *Discrete Cosine Transform - Algorithms, Advantages and Applications.* Academic Press, 1990.

[36] O. Rioul and M. Vetterli. Wavelets and Signal Processing. *IEEE Signal Processing Magazine*, pages 14–37, oct 1991.

[37] M. J. Smith and T. P. Barnwell. Exact Reconstruction techniques for tree-structured subband coders. *IEEE Trans. on ASSP*, 34:434–441, 1986.

[38] A. K. Soman and P. P. Vaidyanathan. Paraunitary filter banks and wavelet packets. In *Proceedings of ICASSP, San Francisco, CA*, volume 4, pages IV–397–IV–400. IEEE, 1992.

[39] P. Steffen. Closed Form Derivation of Orthogonal Wavelet Bases for Arbitrary Integer Scales. Technical report, University of Erlangen-Nürnberg, Germany, 1992.

[40] G. Strang. The Optimal Coefficients in Daubechies Wavelets. *Physica D*, 60:239–244, 1992.

[41] P. P. Vaidyanathan. Quadrature mirror filter banks, m-band extensions and perfect-reconstruction techniques. *IEEE ASSP Magazine*, pages 4–20, 1987.

[42] P. P. Vaidyanathan. Improved Technique for The Design of Perfect Reconstruction FIR QMF Banks with Lossless Polyphase Matrices. *IEEE Trans. on ASSP*, 37(7):1042–1056, 1989.

[43] P. P. Vaidyanathan. Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial. *IEEE Proceedings*, 78:56–93, January 1990.

[44] P. P. Vaidyanathan. *Mutirate Systems and Filter Banks.* Prentice Hall, 1992.

[45] P. P. Vaidyanathan and Z. Doganata. The Role of Lossless Systems in Modern Digital Signal Processing. *IEEE Trans. on Education*, 32:181–197, August 1989.

[46] M. Vetterli and Didier Le Gall. Perfect Reconstruction FIR Filter Banks : Some Results and Properties. *IEEE Trans. on ASSP*, 37(7):1057–1071, 1989.

[47] M. Vetterli and C. Herley. Wavelets and Filter Banks: Relationships and New Results. In *Proceedings of the ICASSP '90 Albuquerque, NM.* IEEE, May 1990.

[48] M. Vetterli and C. Herley. Wavelets and Filter Banks : Theory and Design. *IEEE, Trans. on ASSP*, pages 2207–2232, Sept 1992.

[49] M. J. Vetterli. Multirate Filter Banks. *IEEE Trans. on ASSP*, 35:356–372, 1987.

[50] R. M. Young. *An Introduction to Non-Harmonic Fourier Series.* Academic Press, 1980.

[51] W. R. Zettler, J. Huffman, and D. Linden. The Application of Compactly Supported Wavelets to Image Compression. In *Proceedings of SPIE*, volume 1244, pages 150–160, 1990.

[52] H. Zou and A. H. Tewfik. Discrete Orthogonal M-Band Wavelet Decompositions. In *Proceedings of ICASSP, San Francisco, CA*, volume 4, pages IV–605–IV–608. IEEE, 1992.