# A Pseudo-Bertrand Distribution for Time-Scale Analysis

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**Abstract**

Using the pseudo-Wigner time-frequency distribution as a guide, we derive two new time-scale representations, the pseudo-Bertrand and the smoothed pseudo-Bertrand distributions. Unlike the Bertrand distribution, these representations support efficient online operation at the same computational cost as the continuous wavelet transform. Moreover, they take advantage of the affine smoothing inherent in the sliding structure of their implementation to suppress cumbersome interference components.

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# 1 Introduction

The time-scale distributions of the affine class [1, 2] have proven to be a powerful alternative to time-frequency distributions for the analysis of the time-varying spectral content of nonstationary signals. In contrast to the time and frequency shift covariance of time-frequency distributions, time-scale distributions exhibit an affine covariance; that is, if $\Omega_x(t,f)$ is the distribution of the signal $x(t)$ then the distribution of the shifted and scaled signal $\frac{1}{\sqrt{\alpha}}\, x\left(\frac{t-t_0}{\alpha}\right)$ becomes $\Omega_x\left(\frac{t-t_0}{\alpha}, \alpha f\right)$. Affine covariance makes these new signal representations natural for a host of applications, including wideband radar and sonar, and self-similar signal analysis.

The continuous wavelet transform[1]

$$D_x(t,f) \;\equiv\; f^{\frac{1}{2}} \int x(\tau)\, \psi^*\left(f(\tau-t)\right) d\tau \;=\; f^{-\frac{1}{2}} \int X(\nu)\, \Psi^*(f^{-1}\nu)\, e^{i2\pi\nu t}\, d\nu \tag{1}$$

and the scalogram [2], its squared magnitude, are certainly the most popular time-scale distributions. However, due to their linear structure, these tools are sensitive to the choice of the wavelet $\psi$ and lack certain desirable properties, such as simultaneously good time and frequency resolution and correct marginals.

To overcome these limitations, a broad class of bilinear distributions covariant to time and scale changes has been developed [1, 2]. Among these representations, the unitary Bertrand distribution [1]

$$B_x(t,f) \;\equiv\; f \int \mu(u)\, X(\lambda(u)f)\, X^*(\lambda(-u)f)\, e^{i2\pi\xi(u)tf}\, du, \tag{2}$$

with $\lambda(u) = \frac{-u}{e^{-u}-1}$, $\xi(u) = \lambda(u) - \lambda(-u) = u$, and $\mu(u) = [\lambda(u)\lambda(-u)]^{\frac{1}{2}} \left(\frac{d\xi(u)}{du}\right)^{\frac{1}{2}}$, plays a central role. In addition to having time-scale covariance, it is unitary, satisfies the frequency and Mellin transform marginals, and localizes on hyperbolic instantaneous frequencies and group delays in the time-frequency plane.

Unfortunately, these desirable properties of the Bertrand distribution are offset by two major practical limitations. First, the entire signal enters into the calculation of the distribution at any point $(t,f)$ in the time-frequency plane, precluding its online operation with long signals. Second, due to its nonlinearity, interference components arise between each pair of signal components, complicating its interpretation [3, 4].

[1]Throughout this paper, we will employ the following notation: the variables $t$ and $f$ correspond to time and frequency, respectively; all integrals run from $-\infty$ to $+\infty$; lower case letters denote time functions and upper case letters denote Fourier transforms. We will also consider only analytic signals, where $X(f) \equiv 0 \;\; \forall\, f < 0$. Usually the wavelet transform is expressed as a function of a time variable $t$ and a scale variable $a$. Here we will use the reparametrization of scale as inverse frequency $a = f_0/f$ suggested in [2] and assume without loss of generality that the center frequency $f_0$ of the wavelet $\psi$ equals 1 Hz.

In this paper, we propose a solution to these problems: a pseudo-Bertrand distribution that not only offers asymptotically the same properties as the Bertrand distribution, but also supports efficient online operation and suppresses troublesome cross-components. Our derivation relies on the strong analogy between time-frequency and time-scale analysis and is inspired by the pseudo-Wigner distribution.

## 2 The Pseudo-Wigner Distribution

While the short-time Fourier transform

$$S_x(t,f) \;\equiv\; \int x(\tau)\, w^*(\tau - t)\, e^{-i2\pi f\tau}\, d\tau,$$

and the spectrogram, its squared magnitude, are natural time-frequency representations, their dependence on the window function $w$ and subsequent lack of simultaneous time and frequency resolution have prompted the development of more advanced bilinear distributions, including the Wigner distribution [5, 6]

$$W_x(t,f) \;\equiv\; \int x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-i2\pi f\tau}\, d\tau. \tag{3}$$

This representation overcomes most of the drawbacks associated with the spectrogram, but because it matches the window to the signal, it suffers from two major limitations of its own. First, it does not support online operation, since its calculation requires the entire signal. Second, its interpretation is complicated by nonlinear interference components.

The pseudo-Wigner distribution [6], a sliding version of the Wigner distribution, results from inserting a window function $h$ into (3)

$$\widetilde{W}_x(t,f) \;\equiv\; \int x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) h(\tau)\, e^{-i2\pi f\tau}\, d\tau. \tag{4}$$

Loosely speaking, this representation is equivalent to the Wigner distribution of the time windowed signal $x(\tau)\sqrt{h(2\tau - t)}$, meaning large amounts of data can be treated online. Alternatively, the pseudo-Wigner distribution can be written in terms of a convolution in frequency of two short-time Fourier transforms computed with window $w(\tau) = \sqrt{h(2\tau)}$ [7]

$$\widetilde{W}_x(t,f) \;=\; \int S_x(t,\nu)\, S_x(t,\, 2f-\nu)\, e^{-i4\pi t(f-\nu)}\, d\nu. \tag{5}$$

Since time windowing acts as a smoothing in the frequency domain, the pseudo-Wigner distribution suppresses the Wigner distribution interference components that oscillate in the frequency direction.

Moreover, time direction smoothing can be implemented by convolving (4) with a second lowpass function $g$

$$\widetilde{W}_x(t,f) \;\equiv\; \int g(u-t)\left[\int x\left(u+\frac{\tau}{2}\right)x^*\left(u-\frac{\tau}{2}\right)h(\tau)\,e^{-i2\pi f\tau}\,d\tau\right]du. \tag{6}$$

The result is called the smoothed pseudo-Wigner distribution.

The spectrogram, Wigner, pseudo-Wigner, and smoothed pseudo-Wigner distributions belong to a large class of time-frequency distributions, referred to as Cohen's class [5], whose elements can be written as

$$C_x(t,f) \;=\; \iint W_x(u,\nu)\,\Phi(u-t,\nu-f)\,du\,d\nu, \tag{7}$$

with $\Phi$ an arbitrary convolutional kernel. Setting $\Phi(u,\nu) = W_w(u,\nu)$ gives the spectrogram, whereas the separable product $\Phi(u,\nu) = h(u)\,G(\nu)$ gives the smoothed pseudo-Wigner distribution.

## 3 A Pseudo-Bertrand Distribution

Using the results of the previous section as a guide, we now introduce a pseudo-Bertrand distribution. By rewriting (2) in the time domain

$$B_x(t,f) \;=\; \int\left[(\lambda(u)f)^{\frac{1}{2}}\int x(\tau)\,e^{-i2\pi\lambda(u)f(\tau-t)}\,d\tau\right]\left[(\lambda(-u)f)^{\frac{1}{2}}\int x(\tau')\,e^{-i2\pi\lambda(-u)f(\tau'-t)}\,d\tau'\right]^* du, \tag{8}$$

it is clear that the value of the Bertrand distribution at any point $(t,f)$ depends on the entire signal $x$. Since online operation requires that we consider the signal only in a sliding time interval, we introduce a window function $h$ in (8), and define

$$\begin{aligned}\widetilde{B}_x(t,f) \;\equiv\; &\int\left[(\lambda(u)f)^{\frac{1}{2}}\int x(\tau)\,h^*\left[\lambda(u)f(\tau-t)\right]e^{-i2\pi\lambda(u)f(\tau-t)}\,d\tau\right]\\ &\cdot\left[(\lambda(-u)f)^{\frac{1}{2}}\int x(\tau')\,h^*\left[\lambda(-u)f(\tau'-t)\right]e^{-i2\pi\lambda(-u)f(\tau'-t)}\,d\tau'\right]^* du.\end{aligned} \tag{9}$$

The dependence of $h$ on the analysis frequency $f$ guarantees $\widetilde{B}_x$ affine covariance to time shifts and scale changes.[2] By analogy to the pseudo-Wigner distribution, we call this new time-scale representation the *pseudo-Bertrand distribution.*

The special structure of the pseudo-Bertrand distribution (9) admits an efficient online implementation. Introducing the bandpass wavelet function $\psi(\tau) = h(\tau)\,e^{i2\pi\tau}$, we can reorder (9) to yield

$$\widetilde{B}_x(t,f) = \int\left[(\lambda(u)f)^{\frac{1}{2}}\int x(\tau)\,\psi^*\left[\lambda(u)f(\tau-t)\right]d\tau\right]\left[(\lambda(-u)f)^{\frac{1}{2}}\int x(\tau')\,\psi^*\left[\lambda(-u)f(\tau'-t)\right]d\tau'\right]^* du$$

[2] Suppressing the $\lambda(\pm u)$ in $h$ in (9) yields an alternate pseudo-Bertrand distribution with identical covariance properties. However this formulation does not appear to admit an efficient implementation. Rioul and Flandrin consider the same covariance requirements in their definition of the affine pseudo-Wigner distribution [2].

$$= \int D_x\left(t, \lambda(u) f\right)\, D_x^*\left(t, \lambda(-u) f\right) du, \tag{10}$$

where $D_x$ is the wavelet transform of (1) computed with wavelet $\psi$. This multiplicative convolution of two wavelet transforms parallels the expression (5) that holds for the pseudo-Wigner distribution. An algorithm to compute the pseudo-Bertrand distribution runs as follows:

1. Compute the wavelet transform $D_x(t, f)$ with wavelet $\psi(\tau) = h(\tau) e^{i2\pi\tau}$. Samples should be spaced uniformly in time and exponentially in frequency.
2. At each time $t$, for a range of $u$, rescale $D_x(t, f)$ to $D_x(t, \lambda(\pm u) f)$ using the Mellin transform [8], which maps scale changes to simple phase shifts. Since the Mellin transform of a function $z(v)$ equals the Fourier transform of $z(e^v)$, a fast Fourier transform (FFT) applied to the exponentially spaced frequency samples of $D_x(t, f)$ implements a fast Mellin transform.
3. At each time $t$, compute the inner product (10) with respect to $u$.

Using a fast algorithm for the wavelet transform [8, 9], the computational cost of this procedure is $O(MN \log M)$ for $N$ time and $M$ frequency samples,[3] which is on the same order as the cost for the spectrogram, pseudo-Wigner distribution, and scalogram.

In addition to being computationally efficient, the pseudo-Bertrand distribution suppresses interference components oscillating in the frequency direction, since the frequency-dependent windowing in (9) acts as a constant-$Q$ frequency smoothing. To suppress interference components oscillating in the time direction, we introduce a proportional-bandwidth time smoothing through a second lowpass function $g$

$$\widetilde{B}_x(t, f) \;=\; \int g(u)\, D_x\left(t,\, f\lambda(u)\right)\, D_x^*\left(t,\, f\lambda(-u)\right) du. \tag{11}$$

We call this new time-scale representation the *smoothed pseudo-Bertrand distribution.*

## 4 Kernel Formulation of the Pseudo-Bertrand Distribution

The pseudo-Bertrand distribution (10) and the smoothed pseudo-Bertrand distribution (11) can be related to the important Bertrand distribution by an affine smoothing of the form

$$\widetilde{B}_x(t, f) \;=\; \iint B_x(\tau, \zeta)\, \Pi\left(\zeta(\tau - t),\, \frac{\zeta}{f}\right) d\tau\, d\zeta, \tag{12}$$

with $\Pi$ a kernel function.[4]

[3] We assume that the length of the wavelet at maximum dilation is of $O(M)$.

[4] Formally all affine covariant distributions can be written in this form [10]; therefore (12) serves as an alternative to the Wigner distribution based formula proposed in [2]. However, to ensure well behaved kernel functions $\Pi$, we should use a variant of the Bertrand distribution (2) with $\mu(u) = \lambda(u)\lambda(-u)$ [10].

To show this, we first take the Fourier transform of (2) with respect to $\gamma = tf$ to obtain the companion expression

$$\int B_x\left(\frac{\gamma}{f}, f\right) e^{-i2\pi\gamma\sigma}\, d\gamma \;=\; f\,\mu(\sigma)\, X(\lambda(\sigma)f)\, X^*(\lambda(-\sigma)f). \tag{13}$$

Now, expanding the wavelet transforms in (11) in the frequency domain

$$\widetilde{B}_x(t,f) \;=\; f^{-1}\!\int g(u)\,\mu^{-1}(u) \iint X(\nu)\, X^*(\omega)\; \Psi^*\!\left(\frac{\nu}{f\lambda(u)}\right) \Psi\!\left(\frac{\omega}{f\lambda(-u)}\right) e^{i2\pi t(\nu-\omega)}\, d\nu\, d\omega\, du$$

and introducing the change of variables

$$\nu = \zeta\lambda(v) \quad , \quad \omega = \zeta\lambda(-v) \quad ; \quad \frac{\partial(\nu,\theta)}{\partial(\zeta,v)} = \zeta\lambda(v)\lambda(-v) = \zeta\mu^2(v)$$

yields

$$\begin{aligned}\widetilde{B}_x(t,f) \;&=\; f^{-1} \iiint g(u)\,\frac{\mu(v)}{\mu(u)}\; [\zeta\,\mu(v)\, X(\lambda(v)\,\zeta)\, X^*(\lambda(-v)\,\zeta)] \\ &\qquad \cdot\, \Psi^*\!\left(\frac{\zeta}{f}\frac{\lambda(v)}{\lambda(u)}\right) \Psi\!\left(\frac{\zeta}{f}\frac{\lambda(-v)}{\lambda(-u)}\right) e^{i2\pi\xi(v)t\zeta}\, d\zeta\, dv\, du.\end{aligned}$$

Using (13) for the term in square brackets and the change of variable $\gamma = \zeta\tau$, we obtain (12) with kernel

$$\Pi(\tau,\zeta) \;=\; \zeta \iint g(u)\,\frac{\mu(v)}{\mu(u)}\, \Psi\!\left(\zeta\frac{\lambda(v)}{\lambda(u)}\right) \Psi^*\!\left(\zeta\frac{\lambda(-v)}{\lambda(-u)}\right) e^{i2\pi\xi(v)\tau\zeta}\, du\, dv. \tag{14}$$

It is interesting to note that setting $g(u) = \delta(u)$ in (11) and (14) identifies the kernel $\Pi = B_\psi$ that in (12) generates the scalogram; that is,[5]

$$|D_x(t,f)|^2 \;=\; \iint B_x(\tau,\zeta)\, B_\psi\!\left(\zeta(\tau-t),\, \frac{\zeta}{f}\right) d\tau\, d\zeta.$$

We emphasize, however, the subtile difference with the spectrogram, which can be obtained from (7) with $\Phi = W_w$ but not from (6) with $g(u) = \delta(u)$.

Figure 1 illustrates several time-scale distributions of a synthetic test signal composed of a Lipschitz singularity followed by three modulated Gaussians. While the Bertrand distribution of Figure 1(a) has excellent time-frequency resolution, it also has copious interference components. The constant-$Q$ frequency smoothing of the pseudo-Bertrand distribution of Figure 1(b) suppresses the interference components that oscillate in the frequency direction without affecting the time resolution of the representation. The proportional-bandwidth time smoothing of the smoothed pseudo-Bertrand distribution of Figure 1(c) suppresses the interference components that oscillate in the time direction. For comparison purposes, in Figure 1(d) we plot the scalogram, which can also be obtained from the Bertrand distribution via an affine smoothing.

[5]The same result can be derived directly, using the unitarity and covariance properties of (2).

## 5 Conclusions

Although the Bertrand distribution has many attractive properties, lack of an efficient implementation has limited its impact on time-varying signal analysis. By overcoming some of its limitations, the pseudo-Bertrand and smoothed pseudo-Bertrand distributions should open up new application areas to this powerful tool. Moreover, since the Bertrand distribution belongs to a more general class of affine Wigner distributions of the form (2) with $\lambda_k(u) = [k(e^{-u} - 1)/(e^{-ku} - 1)]^{1/(k-1)}$ [1], we can extend our methods and construct a class of pseudo-affine Wigner distributions. Members of this class, such as the pseudo-Unterberger and the pseudo-D distributions, gain efficient implementations, although expressions such as (12) may not be analytically tractable [10]. Finally, to tune the pseudo-Bertrand distribution to the local characteristics of the signal, we can adapt the wavelet $\psi$ in the sliding algorithm using the techniques of [11].

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# Figure Caption

Figure 1: Time-scale representations of a synthetic test signal composed of a Lipschitz singularity $|t - t_1|^{-0.15}$ followed by three Gaussian windowed tones. The frequency axis runs from 0.05 to 0.5 cycles/sample. (a) Bertrand distribution. (b) Pseudo-Bertrand distribution computed with a Morlet wavelet of $Q = 6$. (c) Smoothed pseudo-Bertrand distribution computed with the same wavelet as in (b) and a square window $g$ of bandwidth 1.4 that time smooths with bandwidth $b(f) = 1.4f$. (d) Scalogram computed with the same wavelet as in (b) and (c).

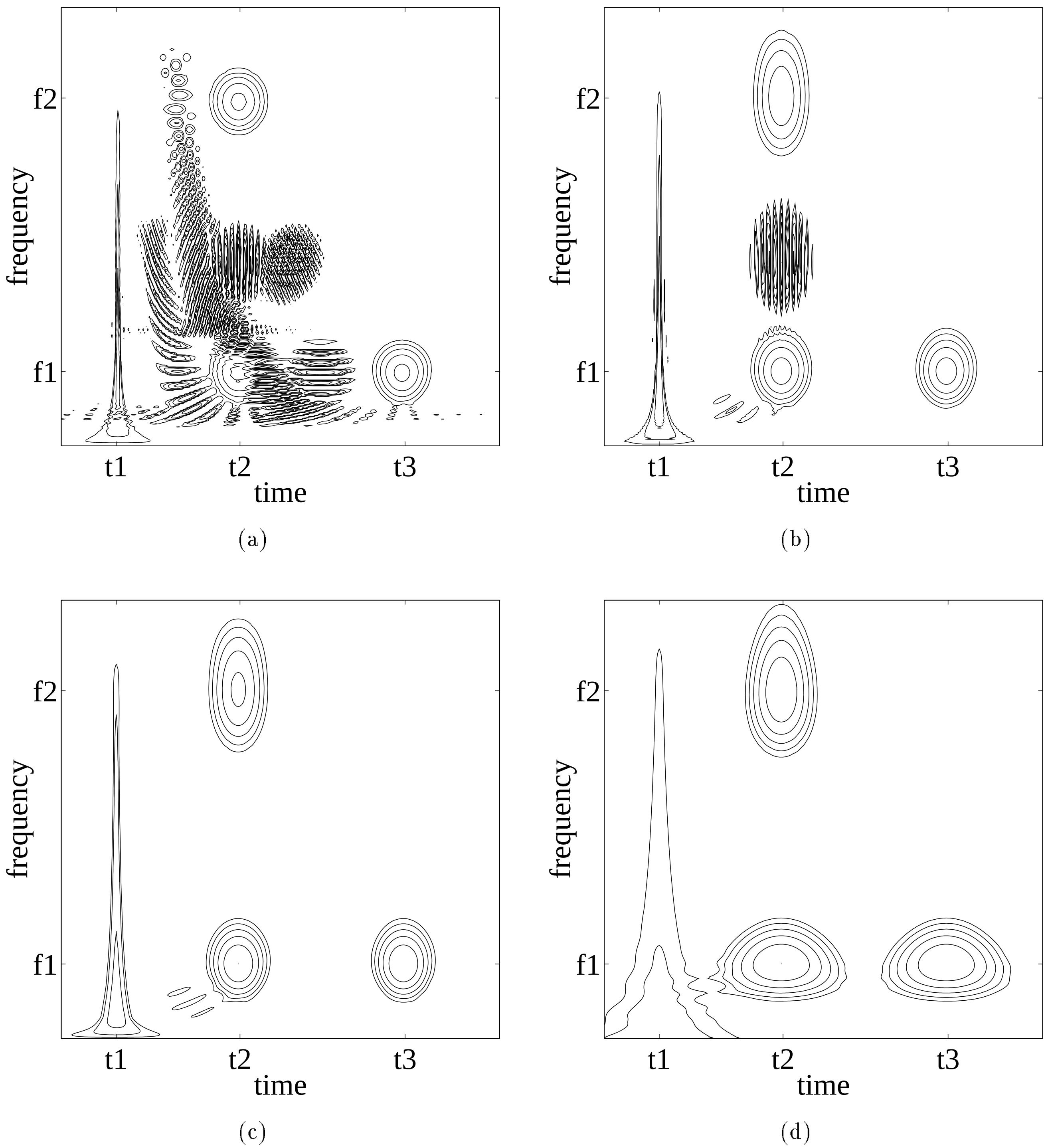


Figure 1: *Time-scale representations of a synthetic test signal composed of a Lipschitz singularity* $|t - t_1|^{-0.15}$ *followed by three Gaussian windowed tones. The frequency axis runs from* $0.05$ *to* $0.5$ *cycles/sample. (a) Bertrand distribution. (b) Pseudo-Bertrand distribution computed with a Morlet wavelet of* $Q = 6$*. (c) Smoothed pseudo-Bertrand distribution computed with the same wavelet as in (b) and a square window* $g$ *of bandwidth* $1.4$ *that time smooths with bandwidth* $b(f) = 1.4f$*. (d) Scalogram computed with the same wavelet as in (b) and (c).*