Wavelet Variations on the Shannon Sampling Theorem

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Foreword

I would like to dedicate this paper to Hans Bremermann on the occasion of this special volume of papers published in his honor. I have known Hans Bremermann since I first got my PhD in several complex variables in 1965 and met Hans at that time in a very memorable meeting in Oberwolfach. Bremermann was extremely warm and cordial to me at the time, and we became friends and met over the years in many places. My wife was also from Bremen, and she and he grew up a few blocks from each other, and we had occasion to see each other at random times in Bremen, and at least once ran into each other on the streets of Bremen, not knowing that the other was even in Europe at the time. Maribel and he were good friends to us over the years, and I visited Berkeley in the summer of 1967 at his urging, and that was an important development in my own career. I too have wandered from my original field of several complex variables, having worked in algebraic geometry, mathematical physics, and more recently mathematics education and applied mathematics and its interaction with engineering. Ths paper here is a collaboration between two undergraduates at Rice at the time of the work (Bray is in graduate school at Stanford in mathematics, and McCormick is in graduate school at Berkeley in physics), and with Xiao-dong Zhou who is a Research Associate here at Rice whom I've worked with for several years now since he got his PhD at Rice with Bob Hardt several years ago. It is, in its own way, symbolic of the interaction which Hans Bremermann exemplified with his work and his spirit.

R. O. Wells, Jr.

1 Introduction

The Shannon Sampling Theorem expresses the fact that a given continuous L^2 function on the real line is determined by the restriction of the function to a specific lattice of points provided that the function is band-limited, i.e., that it has a compactly supported Fourier transform with suitable support. From another point of view, one could say that a suitable function defined on the lattice is extendable in a unique fashion to a continous L^2 function among all such extensions by requiring the band-limited condition on all potential extensions.

In this paper we want to look at two different variations of this result in the wavelet setting. We assume that we have a fixed wavelet basis for $L^2(\mathbf{R})$ consisting of compactly supported wavelets defined by a scaling equation of the form

$$\phi(x) = \sum a_k \phi(2x - k), \tag{1.1}$$

where $\{a_0, \ldots, a_{2g-1}\}$ are scaling coefficients satisfying

$$\sum a_k a_{k+2l} \;=\; 2\delta_{0,l}, \tag{1.2}$$

$$\sum a_k = 2, \tag{1.3}$$

where g is an invariant of the wavelet system called the *genus*. The wavelet function is defined by

$$\psi(x) := \sum b_k \phi(2x - k), \tag{1.4}$$

where $b_k := (-1)^k a_{2g-1-k}$. These are all discussed in the book by Daubechies [3]. In particular, any function $f \in L^2(\mathbf{R})$ has an expansion at a particular scale in terms of both scaling functions and wavelets of the form:

$$f(x) = \sum_{k \in \mathbf{Z}} c_k \phi_k(x) + \sum_{j \in \mathbf{Z}^+, l \in \mathbf{Z}} d_{jl} \psi_{jl}(x), \tag{1.5}$$

where

$$\phi_k(x) := \phi(x - k), \qquad \psi_{jl}(x) := 2^{j/2} \psi(2^j x - k), \tag{1.6}$$

are the translates and rescalings of the scaling and wavelet function which appear in this expansion (here $\mathbf{Z}^+$ denotes the nonnegative integers). We will call the $\{c_k\}$ the *scaling expansion coefficients*, and $\{d_{jl}\}$ the *wavelet expansion coefficients* of the function f.

The variation of the Shannon sampling theorem comes from asking the following question. Given the scaling expansion coefficients $\{c_k\}$ in the expansion (1.5), under what conditions does this determine the function f? In a different manner of speaking, under what conditions do the scaling expansion coefficients determine the wavelet expansion coefficients?

In the context of the Shannon sampling theorem we see that if we identify a sampling of the given function f with its scaling expansion coefficients at a fixed scale (which is done regularly in practical applications of wavelets), then the Shannon sampling theorem tells us that, under suitable hypotheses, knowing the sampling implies that we know the functions at all points, and hence we know all of its coefficients in any orthonormal expansion.

A simple case of an extension would be the trivial extension, i.e., setting $d_{jl} = 0$, for $j \in \mathbf{Z}^+, l \in \mathbf{Z}$. When we identifiy a sampling of a function with its scaling function expansion at a given level, then we are replacing the orginal function with this trivial extension, and there is, in principle, some loss of information (see, e.g., [8] for an error analyis of such an interpolation).

In this paper we give two new extensions which are different from this trivial extension. The first makes the hypothesis that the resulting extension function satisfies a band-limited condtion, and an interpolation formula similar to the Shannon interpolation formula is given. Namely, given a specific choice of scaling expansion coefficients there is a unique choice of an extension $f \in L^2(\mathbf{R})$ such that $\operatorname{supp} \hat{f} \subset (-1/2, 1/2)$. This is discussed in Section 3. The second extension theorem relates to the fact that the scaling and wavelet expansion coefficients are not translation invariant. A variational principle is formulated which minimizes the changes in wavelet expansion coefficients under translations for a given extension of the scaling expansion coefficients $\{c_k\}$. It is shown that there is a unique extension which minimizes this variational integral, and recursive formulas are given for the wavelet expansion coefficients in terms of the scaling expansion coefficients. This second extension always exists, and makes no a priori hypothesis about the extended function. It is not clear at this time how these two unique extensions of a given set of scaling expansion coefficients are related to each other.

In Section 2 is a discussion of the classical Shannon sampling theorem, and it is shown to be a simple consequence of the classical Poisson summation formula from Fourier analysis. This point of view is utilized to formulate and prove a wavelet variation of this same result in Section 3. In Section 4 we find a formulation of the variational principle for extensions of scaling expansion coefficients discussed above, and the existence of a minimizing extension is formulated, the proof of which depends on some delicate estimates which are given in Section 5. Finally, in Section 6 recursive formulas for the wavelet expansion coefficients are derived.

We note in conclusion that other authors have considered wavelet variations of the Shannon sampling theorem, and we refer the reader to [4], and the papers cited therein.

2 The Shannon Sampling Theorem

We have the following version of the Shannon sampling theorem [6] which is convenient for our purposes. We let δ_y be the Dirac delta measure at the point $y \in \mathbf{R}$. We use the convention of the Fourier tranform defined by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbf{R}} e^{-2\pi i x \xi} f(x) dx, \tag{2.7}$$

and we extend this in a natural manner to measures and distributions as given for instance in [7]. Let $I = [-1/2, 1/2]$ be a specific interval of unit length with will play a role in the calculations below.

Theorem 2.1 *Let $f \in L^2(\mathbf{R}) \cap C(\mathbf{R})$ be a continuous square-integrable function. Suppose* $\operatorname{supp} \hat{f} \subset (-1/2, 1/2)$. *Then*

$$g(x) = f(x) \cdot \sum_{k=-\infty}^{\infty} \delta_k \tag{2.8}$$

determines f.

Proof: Let g be the measure given by the formula (2.8), and take the Fourier transform of both sides, using the fact that

$$\mathcal{F}\left(\sum_{k \in \mathbf{Z}} \delta_k \right) = \sum_{k \in \mathbf{Z}} \delta_k, \tag{2.9}$$

which is the Poisson summation formula expressed in this language (see [7]). We obtain

$$\mathcal{F}(g)(\xi) = \mathcal{F}(f)(\xi) * \left(\sum_{k=-\infty}^{\infty} \delta_k \right)$$

Hence we obtain

$$\mathcal{F}(f) = \mathcal{F}(g) \cdot \chi_I$$

and finally we see, by taking the inverse Fourier transform, that f is a function of the scaling expansion coefficients, as given by

$$f = g * \mathcal{F}^{-1}(\chi_I)$$

This last formula is thus a formula for f in terms of g, which is the sampling data, i.e., the values of f at the integer lattice. In particular, the (inverse) Fourier transform of X_I is the *sinc* function, which gives the classical Shannon formula $\square$

3 A Wavelet Varation of the Shannon Sampling Theorem

In this section we want to show a variation on the classical theorem of Shannon.

Theorem 3.1 *Let $f \in L^2(\mathbf{R}) \cap C(\mathbf{R})$ be a continous square-integrable function. Suppose $S := \operatorname{supp}\hat{f} \subset I := (-1/2, 1/2)$, then the scaling expansion coefficients of f determine f.*

Proof: First we note that the scaling expansion coefficients can be expressed in the form

$$\begin{aligned}(f * \check{\phi})(k) &= \int_{-\infty}^{\infty} f(t)\check{\phi}(k-t)dt, \\ &= \int_{-\infty}^{\infty} f(t)\phi(t-k)dt = \langle f \mid \phi_k \rangle,\end{aligned}$$

where we use the standard convention of letting $\check{f}(x) := f(-x)$, and $\langle \cdot \mid \cdot \rangle$ is the L^2 inner product. Hence, we define

$$g = (f * \check{\phi}) \left(\sum_{k=-\infty}^{\infty} \delta_k \right). \tag{3.10}$$

Taking the Fourier transform of (3.10) and using the Poisson summation formula (2.9) as before, we find

$$\begin{aligned}\mathcal{F}(g) &= \mathcal{F}(f * \check{\phi}) * \sum_{k=-\infty}^{\infty} \delta_k \\ &= (\mathcal{F}(f) \cdot \mathcal{F}(\check{\phi})) * \sum_{k=-\infty}^{\infty} \delta_k\end{aligned}$$

Since $\operatorname{supp}\hat{f} \subset S \subset I$,

$$\mathcal{F}(f) \cdot \mathcal{F}(\check{\phi}) = \mathcal{F}(g) \cdot \chi_S$$

which implies that

$$\mathcal{F}(f) = \mathcal{F}(g) \cdot \frac{\chi_S}{\mathcal{F}(\check{\phi})},$$

and thus we obtain that

$$f = g * \mathcal{F}^{-1}\left(\frac{\chi_S}{\mathcal{F}(\phi)}\right). \tag{3.11}$$

□

Remark: Note that the Fourier transform of the scaling function $\mathcal{F}(\phi)$ will vanish at the boundary of I for many wavelet systems, so it is necessary for the support of f to be somewhat smaller to insure the convergence of the formula in (3.11).

4 A Variational Principle: Existence of the Implicit Coefficients

We assume that for some given function f a set of "sampled data points" is given by $\{c_k\}$ where $c_k = \langle f \mid \phi_k \rangle$. The problem we consider is to reconstruct the function f from these scaling expansion coefficients. Let $f_1 := \sum c_k \phi_k$. Clearly, f_1 has the same scaling expansion coefficients as f, but f_1 need not be equal to f at all and may not be a suitable "reconstruction" of f. If we assume that the scale of f corresponds to the scale of ϕ, then we do not expect f to have significant energies in the higher frequency parts of the wavelet spectrum. That is,

$$\sum \langle f \mid \psi_{jk} \rangle^2$$

is small. Here, and in what follows, the sum is over all nonnegative rescalings $j \in \mathbf{Z},\ j \geq 0$, and over all translates $k \in \mathbf{Z}$.

In addition, since the scale of f is independent of its translation, we expect

$$\sum \langle f^\epsilon \mid \psi_{jk} \rangle^2$$

to be small for choices of the translation parameter ϵ ($f^\epsilon(x) = f(x-\epsilon)$). Hence we want the reconstructed function, $\tilde{f}$, to have the minimum amount of small scale information. So we choose $\tilde{f}$ so that

$$\mathcal{H}(\tilde{f}) := \int_0^1 d\epsilon \sum \langle \tilde{f}^\epsilon \mid \psi_{jk} \rangle^2$$

is a minimum. If we consider any variation, g, in span$\{\psi_{jk}\}$ then

$$\frac{\partial}{\partial\alpha}\left(\mathcal{H}(\tilde{f}+\alpha g)\right)|_{\alpha=0} = 0$$

for $\tilde{f}$ a minimum, where $g = \sum \beta_{jk}\psi_{jk}$ and $g^\epsilon = \sum \beta_{jk}\psi^\epsilon_{jk}$. But

$$\mathcal{H}(\tilde{f}+\alpha g) = \int_0^1 d\epsilon \sum \langle \tilde{f}^\epsilon + \alpha g \mid \psi_{jk}\rangle\langle\psi_{jk} \mid \tilde{f}^\epsilon + \alpha g\rangle \tag{4.12}$$

and hence

$$\begin{aligned}\frac{\partial}{\partial\alpha}\left(H(\tilde{f}+\alpha g)\right)|_{\alpha=0} &= \int_0^1 d\epsilon \sum \langle g^\epsilon \mid \psi_{jk}\rangle\langle\psi_{jk} \mid \tilde{f}^\epsilon\rangle \\ &= \sum \beta_{mn}\int_0^1 d\epsilon \sum \langle \psi^\epsilon_{mn} \mid \psi_{jk}\rangle\langle\psi_{jk} \mid \tilde{f}^\epsilon\rangle = 0.\end{aligned}$$

Since the β_{mn} are independent, this gives the conditions on $\tilde{f}^\epsilon$ we are lookig for

$$\int_0^1 d\epsilon \sum \langle \psi^\epsilon_{mn} \mid \psi_{jk}\rangle\langle\psi_{jk} \mid \tilde{f}^\epsilon\rangle = 0, \quad \text{for all } m \in \mathbf{Z}^+, n \in \mathbf{Z}. \tag{4.13}$$

Moreover, if we have two such minima, $\tilde{f}$ and $\tilde{g}$ satisfying (4.13), then, since the functional $\mathcal{H}(\tilde{f})$ is quadratic in $\tilde{f}$, the minimizing points and critical points coincide. Hence if $\tilde{f}$ and $\tilde{g}$ are minima, then $\tilde{f}+\tilde{g}$ is also a minima. Furthermore, if $\tilde{f}, \tilde{g}$ are minima and

$$\langle \tilde{f} \mid \phi_k\rangle = \langle \tilde{g} \mid \phi_k\rangle,$$

then $\tilde{f}-\tilde{g}$ is a minima and

$$\langle \tilde{f}-\tilde{g} \mid \phi_k\rangle = 0.$$

But 0 is also a minima satisfying $\langle 0 \mid \phi_k\rangle = 0$, so

$$\mathcal{H}(\tilde{f}-\tilde{g}) = \int_0^1 d\epsilon \sum \langle \tilde{f}^\epsilon - \tilde{g}^\epsilon \mid \psi_{jk}\rangle^2 = 0$$

which implies that

$$\sum_{\langle} \tilde{f}-\tilde{g} \mid \psi_{jk}\rangle^2 = 0 \Rightarrow \tilde{f}-\tilde{g} = 0 \Rightarrow \tilde{f} = \tilde{g}.$$

We now formulate the following existence theorem. First recall that a wavelet system $\{\phi_k\}$, $\{\psi_{jk}\}$ has *regularity of order* N if

$$\int_{\mathbf{R}} \chi\psi(x)dx = 0, \quad l = 0, \ldots, N.$$

Daubechies showed in [2] that there exist wavelet systems of regularity $N = 0, \ldots, g-1$, where g is the genus of the wavelet system.

Theorem 4.1 *Let $\{\phi_k\}$,$\{\psi_{jk}\}$ be a wavelet system of regularity N at least 2. Let $\{c_k\}$ be a given set of scaling expansion coefficients, then there exists a unique $f \in L^2(\mathbf{R})$ which has $\langle f, \phi_k\rangle = c_k$, and which is a minimum of the functional $\mathcal{H}$.*

The proof of this theorem is given in the next section. We do not know if the regularity hypothesis in Theorem 4.1 is necessary or not.

5 Estimates for the Minimization Extension Theorem

The proof of Theorem 4.1 is given by showing that a minimizing sequence for the functional $\mathcal{H}(f)$ converges weakly in $L^2(\mathbf{R})$. We start with the following lemma.

Lemma 5.1 *If $\{f_i\}$ is an approximating sequence of functions for the minimization of $\mathcal{H}(f)$ subject to the constraint*

$$\langle f \mid \phi_k\rangle = c_k \quad \forall_k \in \mathbf{Z},$$

then $\|f_i\|_{L^2}$ is uniformly bounded.

Proof: Let

$$f_i = \sum c_k\phi_k + \sum b_{jk}\psi_{jk}$$

be a minimizing sequence, where the $\{c_k\}$ are fixed as part of the initial data, and the wavelet expansion coeffients $b_{jk}(i)$ depend on the sequence, and we will write b_{jk} for simplicity, letting the dependence on the sequence index i be implicit. Set now

$$\begin{aligned} P_1(x) &= \sum c_k \phi_k(x), \\ P_2(x) &= \sum b_{jk} \psi_{jk}(x), \end{aligned}$$

and moreover, we see that

$$\|f_i\|_{L^2}^2 = \sum c_k^2 + \sum b_{jk}^2,$$

which we can rewrite in the following form

$$\|f_i\|_{L^2}^2 = \sum c_k^2 + \int_0^1 \mathcal{H}(f_i, \epsilon) d\epsilon + \left[\int_0^1 (\mathcal{H}(f_i, 0) - \mathcal{H}(f_i, \epsilon)\, d\epsilon \right]$$

where

$$\mathcal{H}(f_i, \epsilon) := \sum \langle f_i(x-\epsilon) \mid \psi_{jk} \rangle^2$$

Now we are minimizing $\int_0^1 H(f_i, \epsilon) d\epsilon$ so we may assume that this integral is bounded by Γ, which implies that

$$\|f_i\|_{L^2}^2 \leq \sum c_k^2 + \Gamma + \int_0^1 (H(f_i, 0) - H(f_i, \epsilon))\, d\epsilon$$

Now

$$\begin{aligned} H(f_i, 0) - H(f_i, \epsilon) &= \sum b_{jk}^2 - \sum \langle P_2^\epsilon \mid \psi_{jk} \rangle^2 \\ &\quad - \sum \langle P_1^\epsilon \mid \psi_{jk} \rangle^2 \\ &\quad -2 \sum \langle P_1^\epsilon \mid \psi_{jk} \rangle \langle P_2^\epsilon \mid \psi_{jk} \rangle \end{aligned}$$

because $f_i^\epsilon = P_1^\epsilon + P_2^\epsilon$, so

$$\begin{aligned} \|f_i\|_{L^2}^2 \leq \sum c_k^2 + \Gamma &+ \int_0^1 d\epsilon \left(\sum b_{jk}^2 - \sum \langle P_2^\epsilon \mid \psi_{jk} \rangle^2 \right) \\ &+ \int d\epsilon \sum \langle P_1^\epsilon \mid \psi_{jk} \rangle^2 + 2 \int \left| \sum \langle P_1^\epsilon \mid \psi_{jk} \rangle \langle P_2^\epsilon \mid \psi_{jk} \rangle \right|. \end{aligned}$$

But

$$\sum \langle P_1^\epsilon \mid \psi_{jk} \rangle \langle P_2^\epsilon \mid \psi_{jk} \rangle = \langle P_1^\epsilon \mid P_2^\epsilon \rangle - \sum_l \langle P_1^\epsilon \mid \phi_l \rangle \langle P_2^\epsilon \mid \phi_l \rangle,$$

and

$$\langle P_1^\epsilon \mid P_2^\epsilon \rangle = 0,$$

so

$$\left|\sum_l \langle P_1^\epsilon \mid \phi_l\rangle\langle P_2^\epsilon \mid \phi_l\rangle\right| \geq \left|\sum_{j\in\mathbf{Z}^+,k\in\mathbf{Z}} \langle P_1^\epsilon \mid \psi_{jk}\rangle\langle P_2^\epsilon \mid \psi_{jk}\rangle\right|.$$

By the Cauchy inequality

$$\begin{aligned}\left[\sum_l \langle P_1^\epsilon \mid \phi_l\rangle\langle P_2^\epsilon \mid \phi_l\rangle\right]^2 &\leq \left(\sum_l \langle P_1^\epsilon \mid \phi_l\rangle^2\right)\left(\sum_l \langle P_2^\epsilon \mid \phi_l\rangle^2\right)\\ &\leq \|P_1^\epsilon\|_{L^2}^2\|P_2^\epsilon\|_{L^2}^2 = \left(\sum c_k^2\right)\left(\sum b_{jk}^2\right),\end{aligned}$$

so

$$\int\left|\sum_{j\in\mathbf{Z}^+,k\in\mathbf{Z}} \langle P_1^\epsilon \mid \psi_{jk}\rangle\langle P_2^\epsilon \mid \psi_{jk}\rangle\right| \leq \left(\sum c_k^2\right)^{\frac{1}{2}}\left(\sum b_{jk}^2\right)^{\frac{1}{2}},$$

which implies that

$$\begin{aligned}\|f_i\|_{L^2}^2 \leq 2\sum c_k^2 + \Gamma &+ 2\sqrt{\sum c_k^2}\sqrt{\sum b_{jk}^2}\\ &+ \int_0^1 d\epsilon\left(\sum b_{jk}^2 - \sum\langle P_2^\epsilon \mid \psi_{jk}\rangle^2\right).\end{aligned}$$

Now if we could prove that

$$\int_0^1 d\epsilon \sum b_{jk}^2 - \sum\langle P_2^\epsilon \mid \psi_{jk}\rangle^2 \leq \alpha\sum b_{jk}^2,$$

where $\alpha < 1$ then we would have

$$\sum b_{jk}^2 + \sum c_k^2 = \|f_i\|_{L^2}^2 \leq 2\sum c_k^2 + \Gamma + 2\sqrt{\sum c_k^2}\sqrt{\sum b_{jk}^2} + \alpha\sum b_{jk}^2.$$

Let $\eta^2 = \sum b_{jk}^2$ then

$$(1-\alpha)\eta^2 \leq \sum c_k^2 + \Gamma + 2\sqrt{\sum c_k^2}\cdot\eta$$

which implies that

$$(1-\alpha)\eta^2 - 2\sqrt{\sum c_k^2}\eta - \left(\sum c_k^2 + \Gamma\right) \leq 0,$$

and thus we see that η is bounded .

Lemma 5.2 *Let* $f = \sum b_{jk}\psi_{jk}$, *then*

$$\int_0^1 d\epsilon \left(\sum b_{jk}^2 - \sum \langle P_2^\epsilon \mid \psi_{jk}\rangle^2\right) \leq \alpha \sum b_{jk}^2$$

if and only if

$$\|f * \phi\|_{L^2}^2 \leq \alpha \|f\|_{L^2}^2$$

where

$$\|f * \phi\|_{L^2}^2 = \int \left|\hat{f}\right|^2 \left|\hat{\phi}\right|^2 d\xi$$

Proof: By the assumption on f, we have

$$\begin{aligned}
&\int_0^1 d\epsilon \left(\sum b_{jk}^2 - \sum \langle f^\epsilon, \psi_{jk}\rangle^2\right) \\
&\quad = \int_0^1 d\epsilon \left(\|f^\epsilon\|_{L^2}^2 - \left[\|f^\epsilon\|_{L^2}^2 - \sum \langle f^\epsilon, \phi_k\rangle^2\right]\right) \\
&\quad = \int_0^1 d\epsilon \left(\sum \langle f^\epsilon, \phi_k\rangle^2\right) \\
&\quad = \int_0^1 d\epsilon \sum_k \left[\int_{\mathbf{R}} f(x-\epsilon)\phi(x-k)dx\right]^2 \\
&\quad = \sum_k \int_0^1 d\epsilon \left[\int_{\mathbf{R}} f(x+k-\epsilon)\phi(x)dx\right]^2 \\
&\quad = \sum_k \int_{-k}^{-k+1} dz \left[\int_{\mathbf{R}} f(x-z)\phi(x)dx\right]^2 \\
&\quad = \int_{-\infty}^{\infty} dz \left[(f * \phi)(z)\right]^2 \\
&\quad = \|f * \phi\|_{L^2}^2 = \left\|\widehat{f * \phi}\right\|_{L^2}^2 \\
&\quad = \left\|\hat{f}\hat{\phi}\right\|_{L^2}^2
\end{aligned}$$

□

We know for the wavelets of interest that

$$\left|\hat{\phi}\right|^2 \leq 1$$

and, since these are positive quantities,

$$\int \left|\hat{f}\right|^2 \left|\hat{\phi}\right|^2 d\xi \leq \int \left|\hat{f}\right|^2 g d\xi,$$

where g is a positive real-valued function such that

$$g(x) \geq \left|\hat{\phi}(x)\right|^2.$$

Now let μ be a C^∞ function with support in $(-2, 2)$, $\mu \geq 0$ everywhere and $\mu(\xi) \equiv 1$ on $(-1, 1)$.

Suppose that for some $\epsilon > 0$

$$\int_{-2\epsilon}^{2\epsilon} \left|\hat{f}(\xi)\right|^2 \mu\left(\frac{\chi}{\epsilon}\right) d\xi < \alpha \int \left|\hat{f}(\xi)\right|^2 d\xi,$$

where $\alpha < 1$ (this will be shown below in Lemma 5.3). But on $(\epsilon, \infty) \cup (-\infty, -\epsilon)$ $|\hat{\hat{\phi}}|^2$ is bounded by $\tau < 1$, so take

$$g = \tau + (1 - \tau)\mu\left(\frac{x}{\epsilon}\right),$$

then on $(\epsilon, \infty) \cup (-\infty, -\epsilon)$, $g \geq \tau \geq \left|\hat{\phi}\right|^2$ and on $(-\epsilon, \epsilon)$ $\mu\left(\frac{x}{\epsilon}\right) \equiv 1$, and therefore, on this interval, $g \equiv 1 \geq \left|\hat{\phi}\right|^2$. Hence $\tau + (1 - \tau)\mu(\xi)$ is an upper bound for $|\hat{\phi}|^2$. But

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 g(\xi) d\xi &= \left(\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2\right) \tau d\xi \\ &+ \left(\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 \mu\left(\frac{\xi}{\epsilon}\right)\right) (1 - \tau) d\xi,\end{aligned}$$

and thus we obtain

$$\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 g(\xi) d\xi \leq (\tau + \alpha(1 - \tau)) \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi < 1 \text{ since } \alpha < 1.$$

We now have the following lemma which will complete the proof of the boundedness of the minimizing sequence.

Lemma 5.3 *Suppose*

$$f = \sum b_{jk}\psi_{jk}, \tag{5.14}$$

then, there exists an $\epsilon > 0$ *such that there is a positive* $\alpha < 1$, *such that*

$$\int_{-2\epsilon}^{2\epsilon} \left|\hat{f}(\xi)\right|^2 \mu\left(\frac{\xi}{\epsilon}\right) dx < \alpha \int |\hat{f}(\xi)|^2 dx$$

Proof: From (5.14) it follows that

$$\begin{aligned} \hat{f} &= \sum_j \sum_k b_{jk} 2^{\frac{-j}{2}} e^{\frac{-ik2\pi\xi}{2^j}} \hat{\psi}\left(\frac{\xi}{2^j}\right) \\ &= \sum_j 2^{\frac{-1}{2^j}} \hat{\psi}\left(\frac{\xi}{2^j}\right) \sum_k b_{jk} e^{\frac{-ik2\pi\xi}{2^j}}, \end{aligned}$$

and hence,

$$\left|\hat{f}(x)\right|^2 = \hat{f} \cdot \bar{\hat{f}} = \sum_{j_1,j_2} 2^{\frac{-1}{2}(j_1+j_2)} \hat{\psi}\left(\frac{\xi}{2^{j_1}}\right) \bar{\hat{\psi}}\left(\frac{\xi}{2^{j_2}}\right) \sum_{k_1,k_2} b_{j_1k_1} b_{j_2k_2} e^{i\left(\frac{k_2}{2^{j_2}} - \frac{k_1}{2^{j_1}}\right)2\pi\xi}.$$

We may assume $j_2 \geq j_1$ since this will only introduce a constant, so we have to estimate the integral E given by

$$E := \int_{-2\epsilon}^{2\epsilon} \left(\sum_{j_2 \geq j_1 \geq 0} 2^{\frac{-1}{2}(j_1+j_2)} \hat{\psi}\left(\frac{\xi}{2^{j_1}}\right) \bar{\hat{\psi}}\left(\frac{\xi}{2^{j_2}}\right) \sum_{k_1,k_2} b_{j_1k_1} b_{j_2k_2} e^{i\left(\frac{k_2}{2^{j_2}} - \frac{k_1}{2^{j_1}}\right)2\pi\xi} \right) \mu\left(\frac{\xi}{\epsilon}\right) d\xi.$$

Consider the factor

$$\sum_{k_1,k_2} b_{j_1k_1} b_{j_2k_2} e^{i\left(\frac{k_2}{2^{j_2}} - \frac{k_1}{2^{j_1}}\right)2\pi\xi}, \tag{5.15}$$

and let

$$\beta = k_2 - 2^{j_2-j_1}k_1, \tag{5.16}$$

then (5.15) becomes

$$\sum_{k_1,\beta} b_{j_1k_1} b_{j_2,\beta+2^{j_2-j_1}k_1} \cdot e^{\frac{i}{2^{j_2}}\beta 2\pi\xi}. \tag{5.17}$$

So, by substituting (5.16) into the expression for E, we obtain

$$E = \int_{-2\epsilon}^{2\epsilon} d\xi \mu \left(\frac{\xi}{\epsilon}\right) \cdot \sum_{\substack{j_1, j_2 \\ j_2 \geq j_1}} 2^{\frac{-1}{2}(j_1+j_2)} \sum_{k_1} b_{j_1 k_1} \sum_{\beta} \hat{\psi}\left(\frac{\xi}{2^{j_1}}\right) \hat{\bar{\psi}}\left(\frac{\xi}{2^{j_2}}\right) e^{\frac{i\beta 2\pi\xi}{2^{j_2}}} b_{j_2, \beta+2^{j_2-j_1}k_1}.$$

Now since $\hat{\psi}(0) = 0$ of order N by the moment vanishing property of the wavelet and $\hat{\psi}$ is an analytic function we have

$$\hat{\psi}\left(\frac{\xi}{2^{j_1}}\right) \hat{\bar{\psi}}\left(\frac{\xi}{2^{j_2}}\right) = \frac{1}{2^{(j_1+j_2)N}} h(\xi)\xi^{2N}$$

for some C^∞ function $h(\xi)$.

Finally let

$$N(\beta) = \sum_{k_1} b_{j_2, \beta+2^{j_2-j_1}k_1} \cdot b_{j_1, k_1}$$

then the integral E becomes

$$E = \sum_{\substack{j_1, j_2 \\ j_2 \geq j_1}} 2^{\frac{-1}{2}(j_1+j_2)} \sum_{\beta} \left(\int_{-2\epsilon}^{2\epsilon} \mu\left(\frac{\xi}{\epsilon}\right) \cdot \frac{1}{2^{(j_1+j_2)N}} h(\xi)\xi^2 e^{\frac{i\beta 2\pi\xi}{2^{j_2}}} d\xi \right) N(\beta).$$

Let $r(\xi) = \mu\left(\frac{\xi}{\epsilon}\right) \cdot h(\xi)\xi^{2N}$ then

$$r'(\xi) = \frac{1}{\epsilon}\mu'\left(\frac{\xi}{\epsilon}\right) h(\xi)\xi^{2N} + \mu\left(\frac{\xi}{\epsilon}\right) h'(\xi)\xi^{2N} + 2N\mu\left(\frac{\xi}{\epsilon}\right) h(\xi)\xi^{2N-1}$$

and

$$\begin{aligned} r'' \quad = \quad & \frac{1}{\epsilon^2}\mu''\left(\frac{\xi}{\epsilon}\right) h(\xi)\xi^{2N} + \frac{1}{\epsilon}\mu'\left(\frac{\xi}{\epsilon}\right) h'(\xi)\xi^{2N} \\ & + \frac{1}{\epsilon}\mu'\left(\frac{\xi}{\epsilon}\right) h'(\xi)(\xi)^{2N} + \mu\left(\frac{\xi}{\epsilon}\right) h''(\xi)\xi^{2N} \\ & + \frac{2N}{\epsilon}\mu'\left(\frac{\xi}{\epsilon}\right) h^(\xi)\xi^{2N-1} + 2N\mu\left(\frac{\xi}{\epsilon}\right) h'(\xi)^{2N-1} \\ & + 2N(2N-1)\mu\left(\frac{\xi}{\epsilon}\right) h(\xi)\xi^{2N-2} \end{aligned}$$

and thus $r''(\xi)$ is uniformly bounded on $(-2\epsilon, 2\epsilon)$ since every $\frac{1}{\epsilon^2}$ has a ξ^2 and every $\frac{1}{\epsilon}$ has a ξ factor. Thus $r''(\xi)$ is bounded by some constant C which is also independent of f_i.

This is important because

$$\int_{-2\epsilon}^{2\epsilon} \mu\left(\frac{\xi}{\epsilon}\right) h(\xi)\xi^{2N} e^{\frac{i\beta 2\pi\xi}{2^{j_2}}} = \int_{-2\epsilon}^{2\epsilon} r''(\xi)\frac{(2^{j_2})^2}{\beta^2(2\pi i)^2} e^{\frac{i\beta 2\pi\xi}{2^{j_2}}},$$

and hence we have the estimates

$$\begin{aligned}\left|\sum_\beta \left(\int_{-2\epsilon}^{2\epsilon} r(\xi) e^{\frac{i\beta 2\pi\xi}{2^{j_2}}} d\xi\right) N(\beta)\right| &\leq \sum_\beta |N(\beta)| \cdot \frac{2^{2j_2}}{4\pi^2\beta^2}\left|\int_{-2\epsilon}^{2\epsilon} r''(\xi) e^{\frac{i\beta 2\pi\xi}{2^{2j}}}\right| \\ &\leq \sum_\beta |N(\beta)| \frac{2^{2j_2}}{\beta^2} \cdot C \cdot \epsilon =^{2^{2j_2}} C\left(\sum_\beta \frac{|N(\beta)|}{\beta^2}\right)\epsilon.\end{aligned}$$

But

$$N(\beta) = \sum_{k_1} b_{j_2, \beta+2^{j_2-j_1}k_1} \cdot b_{j_1,k_1}$$

so by the Cauchy-Schwarz inequality we have

$$\begin{aligned}N(\beta)^2 &\leq \left(\sum_{k_1} b^2_{j_2,\beta+2^{j_2-j_1}k_1}\right)\left(\sum_{k_1} b^2_{j_1,k_1}\right) \\ &\leq \left(\sum b^2_{jk}\right)\left(\sum b^2_{jk}\right) = \left(\sum b^2_{jk}\right)^2\end{aligned}$$

and hence we obtain

$$|N(\beta)| \leq \sum b^2_{jk}.$$

Consequently,

$$\begin{aligned}\sum_\beta \frac{|N(\beta)|}{\beta^2} &\leq \sum b^2_{j,k} \sum_\beta \frac{1}{\beta^2}, \\ &\leq C \cdot \sum_\beta \frac{1}{\beta^2},\end{aligned}$$

and thus we are left with

$$\begin{aligned}&\sum_{0\leq j_2 \leq j_1} 2^{\frac{-1}{2}(j_1+j_2)} \cdot 2^{-(j_1+j_2)N} \cdot 2^{2j_2}\left(\sum_{j\in\mathbf{Z}^+, k\in\mathbf{Z}} b^2_{jk}\right) \cdot C_2 \cdot \epsilon \\ &\leq C_3 \cdot \epsilon \cdot \sum b^2_{jk}.\end{aligned}$$

Thus we obtain
$$\int_{-2\epsilon}^{2\epsilon} \left|\hat{F}(x)\right|^2 \mu\left(\frac{\chi}{\epsilon}\right) < \epsilon \cdot C_3 \cdot \sum b_{jk}^2,$$
and we can choose $\epsilon = \frac{1}{2C_3}$ and obtain
$$\int_{-2\epsilon}^{2\epsilon} \left|\hat{F}(x)\right|^2 \mu\left(\frac{\chi}{\epsilon}\right) \leq \frac{1}{2} \int \left|\hat{F}(x)\right|^2$$
as required.
Proof of Theorem 4.1: Let
$$V_c := \left\{ f \in L^2(c) : \quad \langle f, \phi_k \rangle = c_k, \quad k \in \mathbf{Z} \right\}$$
Then
$$\inf_{f \in V_c} H(f) \leq H\left(\sum_k c_k \phi_k\right) < c + \infty.$$
Choosing $\{f_k\}$ to be the minimization sequence, then
$$\sup_k \|f_k\|_{L^2} < c + \infty.$$
Hence there exists a subsequence $\{f_{k_l}\}_{l=1}^{\infty}$ and $f \in L^2$, such that $f_{k_l} \to f$ weakly in L^2, as $l \to \infty$. Fatou's lemma implies $f \in V_c$ and
$$H(f) = \inf_{g \in V_c} H(g).$$

6 Computing the Wavelet Expansion Coefficients in Terms of the Scaling Expansion Coefficients

In this section we will show that the unique choice of extension described in the previous two sections can be given explicity by an iterative algorithm.

The minimization condition requires
$$\int_0^1 d\epsilon \sum \langle \psi_{mn}^{\epsilon} \mid \psi_{jk} \rangle \langle \psi_{jk} \mid \tilde{f}_{\epsilon} \rangle = 0. \tag{6.18}$$
From linearity we may choose $\{c_k\}$ so that $c_k = \delta_{0,k}$. So $\tilde{f} = \tilde{\phi}$.

Expanding the condition (6.18) above we obtain

$$0 = \langle \psi_{mn} \mid \tilde{\phi} \rangle - \int_0^1 d\epsilon \sum_l \langle \psi_{mn}^\epsilon \mid \phi_l \rangle \langle \phi_l \mid \tilde{\phi}_\epsilon \rangle,$$

and hence

$$\begin{aligned} b_{mn} &= \int_0^1 d\epsilon \sum_l \langle \psi_{mn}^\epsilon \mid \phi_l \rangle \langle \phi_l \mid \phi \rangle \\ &\quad + \sum b_{jk} \int_0^1 d\epsilon \sum_l \langle \psi_{mn}^\epsilon \mid \phi_l \rangle \langle \phi_l \mid \psi_{jk} \rangle. \end{aligned}$$

Once the integrals are evaluated, $\{b_{jk}\}$ may be found by diagonalization. The integrals may be reduced to linear combinations of

$$\Lambda_{m,n} \equiv \int_0^1 d\epsilon \langle \phi^\epsilon \mid \phi_m \rangle \langle \phi_n \mid \phi^\epsilon \rangle$$

by the scaling equation (1.1). Using the scaling equation again,

$$\Lambda_{m,n} = \sum_{i_1,i_2,i_3,i_4} \frac{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}}{8} \left[\Lambda_{2m+i_2-i_1,2n+i_3-i_4} + \Lambda_{2m+i_2-i_1-1,2n+i_3-i_4-1} \right]$$

which allows one to compute the coefficients explicitly. This is similar to the calculation of connection coefficients given in [5], and [1], and we omit any further details here.

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