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ON UPSAMPLING, DOWNSAMPLING AND RATIONAL SAMPLING RATE FILTER BANKS

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Abstract

Recently, solutions to the problem of design of rational sampling rate filter banks in one dimension has been proposed. The ability to interchange the operations of upsampling, downsampling, and filtering plays an important role in these solutions. This paper develops a complete theory for the analysis of arbitrary combinations of upsamplers, downsamplers and filters in multiple dimensions. Though some of the simpler results are well known, the more difficult results concerning swapping upsamplers and downsamplers and variations thereof are new. As an application of this theory, we obtain *algebraic* reductions of the general multidimensional rational sampling rate problem to a multidimensional uniform filter bank problem. However, issues concerning the design of the filters themselves *are not* addressed. In multiple dimensions, upsampling and downsampling operators are determined by integer matrices (as opposed to scalars in one dimension), and the non-commutativity of matrices, makes the problem considerably more difficult. Cascades of upsamplers and downsamplers in one dimension are easy to analyze. The new results for the analysis of multidimensional upsampling and downsampling operators are derived using the Aryabhatta/Bezout identity over integer matrices as a fundamental tool. A number of new results in the theory of integer matrices that a relevant to the filter bank problem are also developed. Special cases of some of the results pertaining to the commutativity of upsamplers/downsamplers have been obtained in parallel by several authors.

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1 Introduction

In multirate Digital Signal Processing and Wavelet Theory the topic of perfect reconstruction filter banks has received widespread attention. The theory of uniform band (or integer sampling rate) filter banks in the one dimensional case is fairly well understood [20, 17]. However, not much in the form of a well-developed theory is known in the multidimensional case of uniform band (or integer matrix sampling rate) filter banks [21, 13]. Moreover, in the multidimensional case, there has not been any coherent attempt to develop a complete set of tools for the analysis of arbitrary combinations of upsamplers, downsamplers and filters. The real difficultly has been that in multidimensions the operations of upsampling and downsampling correspond to integer matrices, and the non-commutativity of matrices precludes straightforward extensions of the 1-d results. This paper overcomes this difficulty and gives extensions of all the results known in the 1-d case to the multidimensional case. We describe the algebraic operations of upsampling and downsampling in multiple dimensions and using the theory of integer matrices, the Aryabhatta/Bezout identity in particular, we develop a complete set of tools for the analysis of multidimensional filter banks.

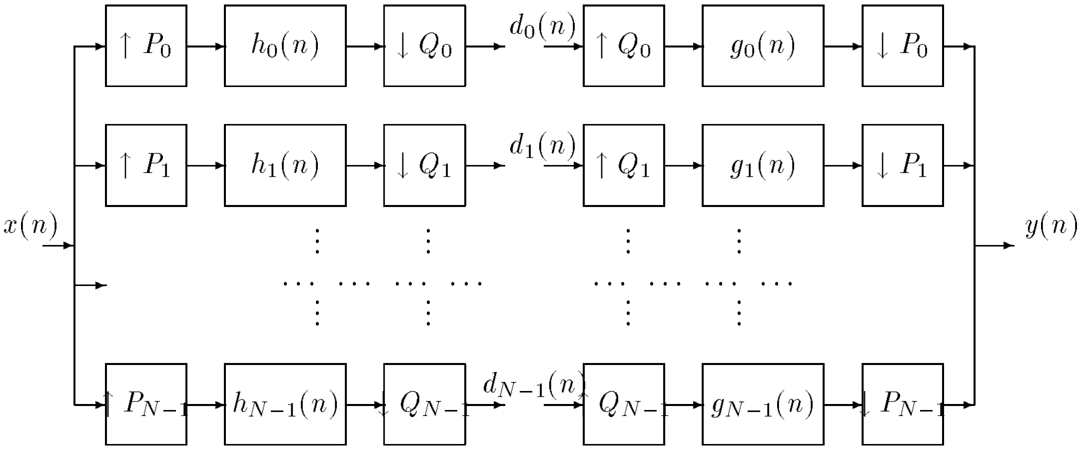

Figure 1: Multidimensional Rational Sampling Rate Filter Bank

Recently, the nonuniform band (or rational sampling rate) filter bank has been the subject of investigation, mainly in one dimension [12, 14, 8, 4]. The general rational filter bank problem in 1-dimension, is given in Fig. 1. There have been two approaches to this problem. The first is the time-domain approach to the problem, where all the constraints for the filter bank to give rise to perfect reconstruction are written in terms of the filter coefficients $h_i(n)$ and $g_i(n)$, and an error function which is a quadratic in the filter coefficients is minimized. This constrained optimization approach to the problem is given in [14]. The second approach, is an algebraic approach where two

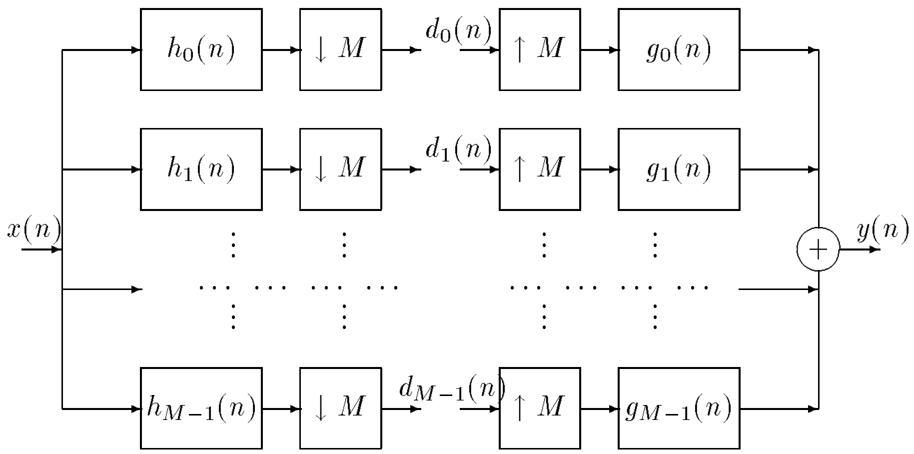

Figure 2: Uniform Sampling Rate Filter Bank

simple algebraic identities involving upsamplers, downsamplers and filters are used to convert the general rational sampling rate problem into the integer sampling rate problem shown in Fig. 2.

As an application of the general theory developed in this paper, we show how the multidimensional rational sampling rate problem can be algebraically reduced into an integer sampling rate problem. The technique adopted is similar to the one dimensional approach in [12]. Moreover, the multidimensional derivation, given the theory developed, is much simpler than the derivations developed in that paper. However, we have not addressed all issues regarding the design of the the rational sampling rate filter bank problem as a uniform sampling rate filter bank problem. This is an open problem.

The problem of commuting upsamplers and downsamplers has been an open problem for some time. In the case of two-dimensions, using a special representation of 2×2 integer matrices generating a given lattice, a necessary and sufficient condition for the commutation of upsamplers and downsamplers has been obtained [11]. We show that our results for the multidimensional case are equivalent to these results in the 2-d case. Moreover, it immediately follows that the two dimensional result in that paper readily generalizes to multidimensional case.

The organization of the paper will be as follows. Section 2 will describe the notational preliminaries. Section 3 will discuss the theory of integer matrices. While most of the results in this section follow from the fact that integer matrices form a *principal ideal domain*, some of the results are new. The results are derived by noticing this fact and imitating the proofs in [9] for polynomial matrices. Some facts about *least common left multiples* and *least common right multiples* of integer matrices that are used in the rational filter bank reduction are also derived. Section 4 will be an overview of lattice theory as it relates to multidimensional upsampling and downsampling. We state and prove the Representatives' Mapping Theorem, and some of its interesting corollaries. Section 5 discusses the fundamental result on on swapping upsamplers and downsamplers, The Swapping Theorem. Section 6 describes how filters can be interchanged with up/downsamplers. Though some of the results in this section are well known, we develop it in a systematic way that gives insight which is helpful in obtaining new results. One such new result is the upsampler-delay-downsamper identity, yet another consequence of of the Aryabhatta/Bezout identity. Section 7 describes the application of the theory to reduce the multidimensional rational sampling rate problem to an integer sampling rate problem. Two algebraic identities are introduced, and used to obtain the final reduction. Finally, in Section 8 we show that our Swapping theorem is a far-reaching generalization of Kovacevic' Theorem [11] on the commutation of upsamplers and downsamplers in two dimensions. In summary, this paper is a tribute to the Aryabhatta/Bezout identity, and its many exquisite consequences.

2 Notational Preliminaries

All through this work, sequences in $\mathbf{Z}^d$ will be denoted as $x(n)$, $y(n)$,..., where $n = (n_1, n_2, \ldots, n_d)^T$. Traditionally in signal processing vectors and matrices are represented in boldface. We prefer *not* to use boldface notation because, this makes the notation independent of dimensions, and moreover makes transparent the relationship between the one and multidimensional cases. Moreover, boldface letters give an *aura* to vectors and matrices that they probably do not deserve. For vectors x and y, we define x^y to be the scalar given by $x^y = \prod_{i=1}^{d} x_i^{y_i}$. For a vector x and a square matrix $M = \begin{bmatrix} m_1 & m_2 & \ldots & m_d \end{bmatrix}$, we define x^M to be the vector (of the same type as x) given by $x^M = (x^{m_1}, x^{m_2}, \ldots, x^{m_d})$. The $\mathcal{Z}$-transform and Fourier transform of a sequence $x(n)$ are functions of d complex variables, $z = (z_1, z_2, \ldots, z_d)^T$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_d)^T$ defined by

$$X(z) = \sum_{n \in \mathbf{Z}^d} x(n) z^{-n} \quad \text{and} \quad X(\omega) = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath \omega^T n}.$$

where $\imath = \sqrt{-1}$. For a matrix M, $|M|$ will denote the absolute value of the determinant of M.

3 Integer Matrices

In multidimensional up/downsampling integer matrices play the same role that the integers play in one dimensional up/downsampling. The crucial properties of the integers useful for the analysis of general 1-d systems are abstracted and analogues are found for integer matrices giving a framework for the analysis of multidimensional multirate systems.

For $m, n \in \mathbf{Z}$, let $r = \gcd(m, n)$, and $l = \mathrm{lcm}(m, n)$. The gcd and lcm are unique only modulo multiplication by ± 1, the units of the ring $\mathbf{Z}$. There exist coprime integers c and d ($\gcd(c, d) = 1$) such that $l = cm = dn$. Similarly there exist coprime integers a and b such that $r = am + bn$ [2]. Unlike c and d (which are unique modulo ± 1), a and b are non-unique: a could be replaced by $a - kn$ and b by $b + kn$ for all $k \in \mathbf{Z}$. Also, the lcm can be written as $l = \pm mn/r$ which implies that $ad + bc = \mp 1$. These facts are succintly put in the form

$$\begin{bmatrix} a - kn & b + km \\ c & -d \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} \stackrel{\text{def}}{=} U \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}, \tag{1}$$

where $\det U = \mp 1$ and $k \in \mathbf{Z}$. That $\det U = \pm 1$ ensures that $\gcd(a, b) = \gcd(c, d) = 1$.

An analog of Eqn. 1 in the form of the Aryabhatta/Bezout identity will *the* fundamental tool in this paper. Integer matrices with integer inverses are said to be *unimodular*. For any matrix A, $A^{-1} = \frac{1}{\det A} \mathrm{adj} A$. Therefore U is unimodular iff $\det U = \pm 1$. Unimodular matrices can always be represented as a product of *elementary* unimodular matrices ([9]) corresponding to *elementary* row/column operations on integer matrices defined as follows: (1) Permutation of two rows/columns, (2) Replacement of a row/column by the sum of itself and an integer multiple of any other row/column, and (3) Negation of any row/column. Using these unimodular operations one can write every integer matrix in three canonical forms [9]:

Lemma 1 *Every integer matrix M can be written in the form*

Left Hermite Form $M = UR$, *U unimodular, lower-triangular and R integer upper triangular.*

Right Hermite Form $M = LV$, *V unimodular, upper-triangular and L integer lower triangular.*

Smith Normal Form $M = U\Sigma V$, Σ *diagonal and* U *and* V *as above. Moreover, the diagonal elements of* Σ, *say* λ_i, *satisfy* $\lambda_{i+1}|\lambda_i$.

Proof: Use the Euclidean algorithm with elementary row operations for the Left Hermite form, elementary column operations for the Right Hermite form and both for the Smith form. □

Remark: Gaussian elimination of real/complex matrices lead to LU decomposition of matrices. The Euclidean algorithm is essentially the same except that the elementary operations are constrained to be over the integers, and this gives the unimodular structure to U and V respectively.

3.1 Divisors and Multiples of Integer Matrices

Let $M = LR$. R is a *right divisor* of M while L is a *left divisor* of M. Also M is a *left multiple* of R and a *right multiple* of L. R is a *common right divisor* of M and N (with the same number of columns) if there exist integer matrices, $\widehat{M}$ and $\widehat{N}$ such that $M = \widehat{M}R$ and $N = \widehat{N}R$. M is a *left common multiple* of R_1 and R_2 if there exists matrices L_1 and L_2 such that $M = L_1R_1$ and $M = L_2R_2$. In this case M is also called the *right common multiple* of L_1 and L_2. A *common left divisor* for integer matrices (with the same number of rows) is defined similarly.

If R is a right divisor of M, R^T is a left divisor of M^T. Also since R being a right divisor is the same as M being a left multiple, M being a left multiple of R is the same as M^T being a right multiple of R^T. Hence it is necessary only to know about right divisors and left multiples.

GCRDs/GCLDs and LCLMs/LCRMs

Definition 1 *A common right divisor,* R*, of* M *and* N *is* a *greatest common right divisor (gcrd) if, for every other common right divisor* $\widehat{R}$*, there exists an integer matrix* W *such that* $R = W\widehat{R}$.

Definition 2 *A common left multiple* M *of* R_1 *and* R_2 *is* a *least common left multiple (lclm) if, for every other left multiple* $\widehat{M}$*, there exists an integer matrix* W *such that* $\widehat{M} = WM$.

Definition 3 *If the gcrd of* M *and* N *is unimodular, then the matrices are said to be right coprime.*

Construction of a GCRD and an LCLM [9]

A gcrd of M and N is constructed by reducing $\begin{bmatrix} M^T & N^T \end{bmatrix}^T$ to its left Hermite form $\begin{bmatrix} R^T & 0 \end{bmatrix}^T$. Then R is a gcrd of M and N. To see this let U be the unimodular matrix involved in the reduction and let $V = U^{-1}$. Then

$$U \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{2}$$

and

$$V \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}. \tag{3}$$

R is a common right divisor since $M = V_{11}R$ and $N = V_{21}R$. Let R_1 be any other common right divisor (i.e., $M = M_1R_1$ and $N = N_1R_1$). Since $R = U_{11}M + U_{12}N$ it follows that

$$R = [U_{11}M_1 + U_{12}N_1]R_1, \tag{4}$$

implying that R is a gcrd. Moreover, if we define

$$L = U_{21}M = -U_{22}N, \tag{5}$$

L is an lclm of M and N. From Definition 2, L is an lclm iff U_{21} and U_{22} are left coprime. This is indeed the case, since U is unimodular (as will be obvious when we develop a characterization of coprimeness).

Gcrds are non-unique since the Hermite form is not unique. However, if one gcrd is non-singular, then all other gcrds are related by left multiplication by a unimodular matrix.

Lemma 2 (Characterization of Right Coprimeness) *M and N are right coprime iff*

1. *There exists integer matrices X and Y such that $XM + YN = I$.*
2. *The matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ is left invertible.*

Proof: Clearly the two statements are equivalent. Since a gcrd is unimodular from Eqn. 4 it follows that $I = R^{-1}U_{11}M + R^{-1}U_{12}N$. Take $X = R^{-1}U_{11}$ and $Y = R^{-1}U_{12}$. □

The first result is the Bezout identity [9]. The second result says that says that coprimeness is an invertibility property. If we take *any* left invertible (over the integers) integer matrix, every subpartition of the rows of the matrix gives two right coprime matrices. Coprimeness is a collective property of the rows of M and N. By transposition it follows that X^T and Y^T are right coprime (i.e., X and Y are left coprime).

From the unimodularity of U and V, $U_{21}V_{12} + U_{22}V_{22} = I$. Therefore (from Lemma 2) U_{21} and U_{22} are left coprime, establishing that L is indeed an lclm of M and N.

The Aryabhatta/Bezout identity compactly encodes all this information:

Lemma 3 (Aryabhatta/Bezout Identity) *M and N are right coprime iff there exist integer matrices $\tilde{M}$, $\tilde{N}$, X, Y, $\tilde{X}$ and $\tilde{Y}$ such that*

$$\begin{bmatrix} \tilde{Y} & \tilde{X} \\ \tilde{N} & -\tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & -Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6}$$

Proof: Similar to the proof for polynomial matrices in [9]. □

Example 1 Consider the following nonsingular matrices M and N ($\det M = 6$, and $\det N = 8$):

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix}.$$

A gcrd of M and N: With

$$D_r = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} -2 & 6 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} D_r \text{ and } N = \begin{bmatrix} -2 & 4 & -2 \\ 2 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} D_r.$$

D_r is a gcrd of M and N. Since D_r is not unimodular, M and N are *not* right coprime.

A gcld of M and N: With

$$D_l = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 3 & -1 \end{bmatrix}, \quad M = D_l \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } N = D_l \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -7 & 5 & 1 \end{bmatrix}.$$

D_l is a gcld of M and N. Since D_l is not unimodular M and N are *not* left coprime.

An lclm of M and N: With

$$M_l = \begin{bmatrix} 2 & -2 & -2 \\ 6 & -3 & -3 \\ 0 & 2 & -2 \end{bmatrix}, M_l = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & -4 \\ 0 & 2 & 0 \end{bmatrix} M \text{ and } M_l = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} N.$$

M_l is an lclm of M and N.

An lcrm of M and N: With

$$M_r = \begin{bmatrix} -2 & 0 & 0 \\ -5 & 4 & -3 \\ -14 & 12 & -6 \end{bmatrix}, M_r = M \begin{bmatrix} 4 & -4 & 2 \\ -5 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } M_r = N \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -7 & 5 & -3 \end{bmatrix}.$$

M_r is an lcrm of M and N.

Example 2[Aryabhatta/Bezout Identity] Consider the following right coprime matrices M and N:

$$M = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}.$$

By using the Hermite form one obtains the Aryabhatta/Bezout identity

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 1 & -1 & 4 & 0 \\ 0 & 2 & -4 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

One identifies X, Y, $\tilde{X}$, $\tilde{Y}$, $\tilde{M}$ and $\tilde{N}$ by comparing with Eqn. 6.

GCRD/LCLM of sets of matrices

The concepts of gcrds and lclms can be generalized to the case of many matrices. Lemma 2 tells us that coprimeness is equivalent to the existence of a left inverse for the block matrix formed by stacking the individual matrices one below the other. Computing the gcrd of a set of matrices is the same a computing for just two of them. On the other hand the computation of the lclm is not as straightforward. However, the essential idea is as follows. Given a set of l integer matrices, $M_0, \ldots M_{l-1}$, an lclm of all of them may be obtained as follows: Let $P_1 = \text{lclm}(M_0, M_1)$. Recursively define $P_i = \text{lclm}(P_{i-1}, M_i)$. Then P_{l-1} is an lclm of the entire set of matrices.

4 Fundamental Tools in Multirate Signal Analysis

Filters, upsamplers and downsamplers are the basic building blocks in multirate systems theory. Sampling rate conversion is provided by upsamplers and downsamplers. Filters, besides traditional filtering, also appear in *anti-aliasing* and *image removal* roles. Multirate signal analysis addresses cascades of these building blocks. For one dimensional signals, the necessary tools for multirate signal analysis are well known [15]. Recently many researchers have tried to extend these results to multidimensional signals, with varying degrees of success [11, 5, 4, 7, 6, 10]. This paper obtains a comprehensive set of algebraic tools for the analysis of multidimensional multirate systems using the Aryabhatta/Bezout identity.

4.1 Lattices, Upsampling and Downsampling

In one dimension, if $x(n)$ is the input of a 2-fold downsampler, then the output $y(n)$ is $x(2n)$. Similarly, the output $y(n)$, of a 2-fold upsampler is obtained by interlacing the input sequence $x(n)$ with zeros. The "right" way to think about this is in terms of lattices. Both $x(n)$ and $y(n)$ are defined on a lattice of points, namely $\mathbf{Z}$. The output of a 2-fold downsampler, is the input on a sublattice of $\mathbf{Z}$, namely $2\mathbf{Z}$. The upsampling operator chooses a bigger lattice and embeds the input sequence on a sublattice of this bigger lattice. This viewpoint readily generalizes to multidimensional downsampling and upsampling.

Let $\mathcal{L}$ denote integers in $\mathbb{R}^d$ (i.e., $\mathcal{L} = \mathbf{Z}^d$). For a non-singular matrix M over $\mathbb{R}$, the set

$$\mathcal{L}(M) = \{Mn \mid n \in \mathcal{L}\} = M\mathcal{L} \tag{7}$$

is the *lattice generated* by M. Fig. 3 shows the lattice generated by the matrix $M = \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix}$.

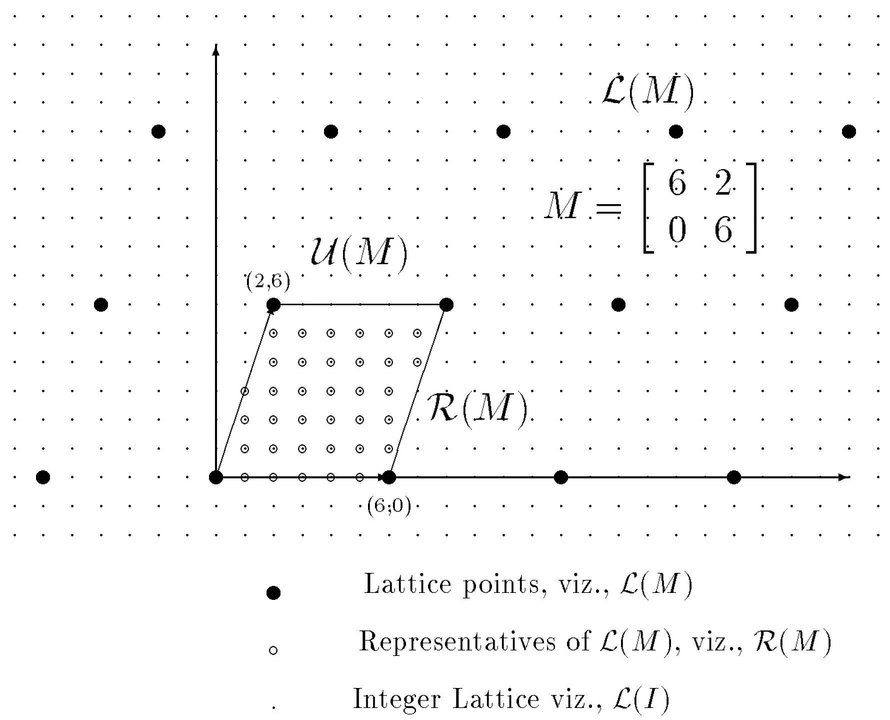

Figure 3: Lattice Generated by M

If $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$, $\mathcal{L}(M_1)$ is a *sublattice* of $\mathcal{L}(M_2)$. The geometric condition of being a sublattice can be algebraically characterized neatly [3]:

Fact 1 *$\mathcal{L}(M_1)$ is a subset of $\mathcal{L}(M_2)$ iff there exists an integer matrix K such that $M_1 = M_2K$.*

Fact 1 implies that $\mathcal{L}(M) \subseteq \mathcal{L} = \mathcal{L}(I)$ iff M is integral (see Fig. 3). In the example above, the number of lattice points per unit volume $1/36 = 1/\,|M|$. This is true for all lattices [3, 18].

The *generator* of a lattice is not *unique*. In Fig. 3

$$M = \begin{bmatrix} 8 & 2 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

is also a generator of same lattice. The generator is unique only up to *right* multiplication by a *unimodular* matrix. If M is unimodular, then the number of lattice points per unit volume is $1/|M| = 1$. Since $\mathcal{L}(M)$ is a sublattice of $\mathcal{L}(I)$ which also has 1 lattice point per unit volume, one would expect that $\mathcal{L}(M) = \mathcal{L}(I)$. More formally, if $MU = N$, and $V = U^{-1}$, from Fact 1 $\mathcal{L}(N) \subseteq \mathcal{L}(M)$. Also since V is unimodular, $M = NV$ and $\mathcal{L}(M) \subseteq \mathcal{L}(N)$. Therefore $\mathcal{L}(M) = \mathcal{L}(N)$.

Let $\mathcal{U}$ denote the unit cube in $\mathbb{R}^d$. That is $\mathcal{U} = \left\{x \in \mathbb{R}^d \mid x_i \in [0,1)\right\} = [0,1)^d$. The *unit cell*, $\mathcal{U}(M)$, of a lattice $\mathcal{L}(M)$ is defined to be the image of $\mathcal{U}$ under M. That is $\mathcal{U}(M) = \left\{Mx \in \mathbb{R}^d \mid x \in \mathcal{U}\right\} = M\mathcal{U}$. Every point in $\mathbb{R}^d$ can be represented uniquely as a linear combination of a point in $\mathcal{U}(M)$ and $\mathcal{L}(M)$.

Lemma 4 *For all $x \in \mathbb{R}^d$ there exists a unique, $\mathcal{LU}$ decomposition to $\mathcal{L}(M)$, $x = x_l + x_u$, with $x_l \in \mathcal{L}(M)$ and $x_u \in \mathcal{U}(M)$*

Proof: By (geometrically) translating the unit cell to any integer in $\mathbb{R}^d$, it is clear that any point $y = M^{-1}x$ in $\mathbb{R}^d$ can be represented uniquely as $l + u$, $l \in \mathcal{L}$ and $u \in \mathcal{U}$. Take $x_l = Ml$ and $x_u = Mu$ to get the result. □

In particular if $x = n$ is an integer, and M is integer non-singular, then both x_l and x_u are integers. In this case the integer x_u is denoted by $n \bmod M$ or $(n)_M$. It turns out that two points in the unit cell cannot differ by a lattice point, as is easily seen from Fig. 3.

Lemma 5 *If $\alpha, \beta \in \mathcal{U}(M)$, then $\alpha - \beta \in \mathcal{L}(M)$ iff $\alpha = \beta$.*

Proof: In the scalar case if $\alpha, \beta \in [0,1)$ implies $\alpha - \beta \in (-1,1)$. The only integral value it can take is 0. Similarly, $\alpha, \beta \in [0,1)^d$ implies $\alpha - \beta \in (-1,1)^d$. The only integer in this volume is 0. This fact in conjunction with Lemma 4 gives the result. □

Any integer in the unit cell of a lattice is called a *representative* of the lattice. The set of all representatives, $\mathcal{R}(M)$, the lattice and its unit cell are relates as follows: $\mathcal{R}(M) = \mathcal{L}(I) \bigcap \mathcal{U}(M)$. There are precisely, $\lceil |M| \rceil$ elements in $\mathcal{R}(M)$. In this paper we mainly deal with integer matrices M for which there are precisely $|M|$ representatives. While discussing filter bank theory, it will be necessary to consider sets similar to $\mathcal{R}(M)$, that we call *generalized* sets of representatives.

Definition 4 *A set $\mathcal{S}(M) \subset \mathbb{R}^d$ of precisely $\lceil |M| \rceil$ points is said to be a generalized set of representatives of the lattice $\mathcal{L}(M)$ if $\mathcal{S}(M)(\bmod M) = \mathcal{R}(M)$.*

For any given $\mathcal{L}(M)$ there are infinitely many generalized sets of representatives. An important property of any generalized set of representatives, $\mathcal{S}(M)$, is that for any two points in $\mathcal{S}(M)$, if the $\mathcal{U}$ parts of their $\mathcal{LU}$ decompositions are equal, the points are the same.

Lemma 6 *Let $k, l \in \mathcal{S}(M)$. Then $k - l \in \mathcal{L}(M)$ iff $k = l$.*

Proof: If $\mathcal{S}(M) = \mathcal{R}(M) \subseteq \mathcal{U}(M)$, then the result follows directly from Lemma 5. Else from the $\mathcal{LU}$ decomposition, $k - l = (k_l - l_l) + (k_u - l_u)$, and therefore, $k - l \in \mathcal{L}(M)$, iff $k_u - l_u \in \mathcal{L}(M)$. But then from Lemma 5, $k_u = l_u$. Since $\mathcal{S}(M)$ is a generalized set of representatives, $k_l = l_l$ and hence $k = l$. $\square$

Example 3 Consider the lattice of integers $\mathbf{Z}$ in $\mathbb{R}$. Any sublattice is given by $M\mathbf{Z}$ for integer M. Assume for simplicity that M is positive. Then unit cell of the lattice is the interval $[0, M)$. Any point $x \in \mathbb{R}$ can be represented uniquely as, $x_l + x_u$, where $x_l = \lfloor \frac{x}{M} \rfloor$, $x_u = [x]$ and $[x]$ denotes the fractional part of x. No two points in $[0, M)$ differ by a lattice point (multiple of M). The representatives of this lattice are the integers $\{0, 1, 2, \ldots, M-1\}$, precisely M of them. The set $\{2, 3, \ldots, M+1\}$ forms *one* set of generalized representatives of $\mathcal{L}(M)$. Two elements of this set also cannot differ by a lattice point.

For M, a non-singular integer matrix the upsampling and downsampling operators are defined as follows (see Fig. 4):

Figure 4: Upsampling and Downsampling by M

Definition 5 *The upsampling operator, $[\uparrow M]$, is defined by*

$$y(n) = [\uparrow M]\, x(n) = \begin{cases} x(M^{-1}n) & \text{for } n \in \mathcal{L}(M) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Definition 6 *The downsampling operator, $[\downarrow M]$, is defined by*

$$y(n) = [\downarrow M]\, x(n) = x(Mn). \tag{9}$$

In general, upsampling is a *reversible* process, while downsampling is an *irreversible* process. The input/output relationships of these operators in the frequency and $\mathcal{Z}$-transform domains are sometimes useful. For a signal $x(n)$, $X(\omega)$ is periodic on the lattice $2\pi\mathcal{L}(I)$ and hence determined uniquely by its values on the unit cell of this lattice, namely $2\pi\mathcal{U}(I)$. However, it is conceivable that $X(\omega)$ is periodic on a lattice bigger than (i.e., is a super-set of) $2\pi\mathcal{L}$. This is precisely what happens after an upsampling. Indeed from Eqn. 8 it follows that

$$Y(z) = [\uparrow M]\, X(z) = \sum_{n \in \mathbf{Z}^d} x(n) z^{-Mn} = X(z^M) \tag{10}$$

and

$$Y(\omega) = [\uparrow M]\, X(\omega) = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath\omega^T M n} = \sum_{n \in \mathbf{Z}^d} x(n) e^{-\imath(M^T\omega)^T n} = X(M^T\omega). \tag{11}$$

Notice that $Y(\omega)$ is periodic on the lattice $2\pi\mathcal{L}(M^{-T})$:

$$Y(\omega + 2\pi M^{-T}k) = X(M^T\omega + 2\pi k) = X(M^T\omega) = Y(\omega). \tag{12}$$

The fundamental period *after interpolation* is not $2\pi\mathcal{U}(I)$, but a subset of it given by $2\pi\mathcal{U}(M^{-T})$. There are $\left|M^T\right| = |M|$ copies of $Y(\omega)$ in $2\pi\mathcal{U}(I)$. Since the response in in each fundamental period is $X(\omega)$, $Y(\omega)$ has $|M|$ *images* of $X(\omega)$ in $2\pi\mathcal{U}(I)$. To remove the imaging caused by upsampling one uses an *image removal* filter whose passband is say, $2\pi\mathcal{U}(M^{-T})$.

Downsampling leads to *aliasing distortion.* For $n \in \mathcal{L}(M)$, $\sum_{k \in \mathcal{R}(M^T)} e^{\imath 2\pi k^T M^{-1} n} = 1$, while for $n \notin \mathcal{L}(M)$, the sum is zero. Using this fact

$$\begin{aligned} Y(z) = [\downarrow M]\, X(z) &= \sum_{n \in \mathbf{Z}^d} x(Mn) z^{-n} \\ &= \sum_{n \in \mathbf{Z}^d} \left\{ \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} x(n) e^{-\imath 2\pi k^T M^{-1} n} \right\} z^{-M^{-1}n} \\ &= \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} X(z^{M^{-1}} e^{-\imath 2\pi k^T M^{-1}}). \end{aligned} \tag{13}$$

Substituting $z = e^{\imath\omega^T}$ we get

$$Y(\omega) = [\downarrow M]\, X(\omega) = \frac{1}{|M|} \sum_{k \in \mathcal{R}(M^T)} X(M^{-T}(\omega - 2\pi k)). \tag{14}$$

$Y(\omega)$ is periodic on $2\pi\mathcal{L}$. At any particular frequency, there are $|M|$ *alias* components of $X(\omega)$ appropriately shifted. It is impossible to reconstruct $X(\omega)$ from $Y(\omega)$ unless all but one of these components is zero. One can ensure this by filtering $X(\omega)$ with an *anti-aliasing* filter prior to downsampling.

We now state and prove a number of facts about integer matrices that will be useful in Section 5. The most notable mathematical result is the Representatives' Mapping Theorem. The multidimensional result is motivated by first considering the one dimensional result, which is well known. Let $M_1\mathbf{Z}$ and $M_2\mathbf{Z}$ be two sublattices of $\mathbf{Z}$ generated by positive integers M_1 and M_2. Then for $i \in \{1,2\}$, the unit cells are given by $[0, M_i)$ and set of representatives are the integers modulo M_i.

$$\mathcal{L}(M_i) = \{\ldots, -M_i, 0, M_i, \ldots\}, \mathcal{U}(M_i) = [0, M_i) \quad \text{and} \quad \mathcal{R}(M_i) = \{0, 1, \ldots, M_i - 1\}.$$

If $\gcd(M_1, M_2) = 1$ it follows that [16],

$$M_2\mathcal{R}(M_1) \bmod M_1 = \mathcal{R}(M_1) \quad \text{and} \quad M_1\mathcal{R}(M_2) \bmod M_2 = \mathcal{R}(M_2). \tag{15}$$

Eqn. 15 states that the representatives of a lattice are mapped back onto itself under multiplicative mapping provided the multiplication is by an integer coprime to the generator of the lattice. The extension of this result to the case of general lattices in $\mathbf{Z}^d$, is the *Representatives' Mapping Theorem.*

Theorem 1 (Representatives' Mapping Theorem) *If M_1 and M_2 are left coprime, there exist right coprime matrices N_1 and N_2 such that,*

$$M_2\mathcal{R}(N_1) \ mod \ M_1 = \mathcal{R}(M_1). \tag{16a}$$

$$M_1\mathcal{R}(N_2) \ mod \ M_2 = \mathcal{R}(M_2). \tag{16b}$$

$$N_1^T\mathcal{R}(M_2^T) \ mod \ N_2^T = \mathcal{R}(N_2^T). \tag{16c}$$

$$N_2^T\mathcal{R}(M_1^T) \ mod \ N_1^T = \mathcal{R}(N_1^T). \tag{16d}$$

Conversely if N_1 and N_2 are right coprime, there exist M_1 and M_2 left coprime, such that Eqns. 16a-16d are true.

Proof: Since M_1 and M_2 are left coprime, by the Aryabhatta/Bezout identity there exist integer matrices P_1, P_2, Q_1, Q_2, N_1, and N_2 such that

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{17}$$

In particular we have the equations:

$$M_1N_2 = M_2N_1. \tag{18a}$$

$$|M_1|\,|N_2| = |M_2|\,|N_1|\,. \tag{18b}$$

$$P_1 N_2 + P_2 N_1 = I. \tag{18c}$$

First consider consider Eqn. 16a. It states that there exists a lattice N_1 such that under the action of M_2, its set of representatives map onto the set of representatives of the lattice M_1.

$\boxed{M_2\mathcal{R}(N_1)\mathrm{mod}M_1 \subseteq \mathcal{R}(M_1)}$ For $n \in \mathbf{Z}^d$, n mod $M_1 \in \mathcal{R}(M_1)$. Therefore for any fixed $k \in \mathcal{R}(N_1)$, M_2k mod $M_1 \in \mathcal{R}(M_1)$.

$\boxed{M_2\mathcal{R}(N_1)\mathrm{mod}M_1 \supseteq \mathcal{R}(M_1)}$ It suffices to show that M_2 induces a one-to-one onto map of $\mathcal{R}(N_1)$ onto $\mathcal{R}(M_1)$.

one-to-one

It is enough to show that for $k, l \in \mathcal{U}(N_1)$, $k = l$ whenever M_2k mod $M_1 = M_2l$ mod M_1. From Lemma 4, there exist $\alpha, \beta \in \mathcal{U}$ such that ($\mathcal{LU}$ decomposition of k and l) $k = N_1\alpha$ and $l = N_1\beta$. Therefore there exists an $n \in \mathbf{Z}^d$ such that

$$\begin{array}{rll} & M_2N_1(\alpha-\beta) = M_1n & \\ \Rightarrow & M_1N_2(\alpha-\beta) = M_1n & \text{Eqn. 18a} \\ \Rightarrow & N_2(\alpha-\beta) = n & M_1 \text{ is invertible} \\ \Rightarrow & P_1N_2(\alpha-\beta) = P_1n & \\ \Rightarrow & (I-P_2N_1)(\alpha-\beta) = P_1n & \text{Eqn. 18c} \\ \Rightarrow & (\alpha-\beta) = P_1n + P_2(l-k) & \\ \Rightarrow & (\alpha-\beta) = 0 & \text{Lemma 5} \end{array}$$

Since $\alpha, \beta \in \mathcal{U}$, this implies $\alpha = \beta$ and hence $k = l$. Also it follows that $|N_1| \leq |M_1|$.

onto

Now it suffices to show that $\left|M_1^T\right| \leq \left|N_1^T\right|$ since then every element of $\mathcal{R}(N_1)$ maps to a unique element of $\mathcal{R}(M_1)$ and conversely. Now using the above arguments and the transposed form of the Aryabhatta/Bezout identity it follows that multiplication by N_2^T, takes $\mathcal{R}(M_1^T)$ in a one-to-one fashion to $\mathcal{R}(N_1^T)$, implying $\left|M_1^T\right| \leq \left|N_1^T\right|$. Therefore, $|M_1| = |N_1|$ and the map induced by M_2 is onto.

We have proved Eqn. 16a. Eqn. 16b follows using the same arguments by replacing M_1 with M_2, N_1 and N_2 etc. Eqn. 16c and Eqn. 16d also can be shown by considering the Aryabhatta/Bezout identity in its transposed (dual) form:

$$\begin{bmatrix} N_2^T & N_1^T \\ Q_2^T & -Q_1^T \end{bmatrix} \begin{bmatrix} P_1^T & M_1^T \\ P_2^T & -M_2^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{19}$$

□

Remark: It also follows that when Q_1 and P_1 (or equivalently Q_2 and P_2) are non-singular we have:

$$P_2\mathcal{R}(Q_1) \bmod P_1 = \mathcal{R}(P_1) \tag{20a}$$

$$P_1\mathcal{R}(Q_2) \bmod P_2 = \mathcal{R}(P_2) \tag{20b}$$

$$Q_1^T\mathcal{R}(P_2^T) \bmod Q_2^T = \mathcal{R}(Q_2^T) \tag{20c}$$

$$Q_2^T\mathcal{R}(P_1^T) \bmod Q_1^T = \mathcal{R}(Q_1^T) \tag{20d}$$

The Representatives' Mapping Theorem has a number of interesting consequences. Two of the results that illuminate the relationship between the scalar case and matrix case are given below:

Corollary 1 *If M_1, M_2 are left coprime and N_1, N_2 are right coprime such that $M_1N_2 = M_2N_1$, then*

$$|M_1| = |N_1| \quad \textit{and} \quad |M_2| = |N_2|\,. \tag{21}$$

Proof: First note that we can always obtain an Aryabhatta/Bezout identity involving the four matrices M_1, M_2, N_1, and N_2. This can be done by taking an Aryabhatta/Bezout corresponding to the N's and tweaking the unimodular matrix U in order to makes the lower block becomes $[M_1 \;\; -M_2]$. Now the proof of Theorem 1 gives the result. □

Commuting integer matrices behave like integers as far as coprimeness is concerned.

Corollary 2 *Non-singular commuting matrices are left coprime iff they are right coprime.*

Proof: Let M_1 and M_2 be left coprime. Then we have an Aryabhatta/Bezout identity with N_1 and N_2 right coprime and $M_1N_2 = M_2N_1 = M$. Since M_1 and M_2 commute the product $\widehat{M}$ is a right common multiple of M_1 and M_2. Therefore there exists a matrix R such that $M_2 = N_2R$ and $M_1 = N_1R$ and hence $\widehat{M} = MR$. Now from Corollary 1, $|M| = |M_1|\,|N_2| = |M_1|\,|M_2| = \left|\widehat{M}\right|$ and therefore R is unimodular. Hence the result follows. $\square$

Remark: Corollary 2 has been obtained recently by other researchers [5]. Their technique does not use the Representatives' Mapping Theorem. Another result that we require later is the following:

Lemma 7 *If $k, l \in \mathcal{S}(M)$ (a generalized set of representatives of $\mathcal{L}(M)$), then*

$$[\downarrow M]\, z^{k-l} = \begin{cases} 1 & \textit{if } k = l \\ 0 & \textit{otherwise.} \end{cases} \tag{22}$$

Proof: Let $x(n) = \delta(n - k + l)$. Then $y(n) = [\uparrow M]\, x(n) = \delta(Mn - k + l)$, and therefore has a value only when $k - l = Mn \in \mathcal{L}(M)$. But from Lemma 6, $k - l \in \mathcal{L}(M)$ iff $k = l$, and therefore, $y(n) = 0$ if $k \neq l$ and $y(n) = \delta(Mn) = \delta(n)$ when $k = l$. Taking the $\mathcal{Z}$-transform on both sides we get the result. $\square$

5 Cascades of Upsamplers and Downsamplers

This section develops the theory for cascades of upsamplers and downsamplers. The results carry over conceptually from the 1-d case. Analysis of general cascades of up/downsamplers can be studied by considering cascades of two up/downsamplers.

5.1 Upsampler-Upsampler

A cascade of upsamplers implies the product of the corresponding upsampling factors in the reverse order as shown in Fig. 5. Indeed from Eqn. 11 one deduces that $[\uparrow M_2]\,[\uparrow M_1] = [\uparrow M_2 M_1]$ since

$$Y(\omega) = Y_1(M_2^T \omega) = X(M_1^T (M_2^T \omega)) = X((M_2 M_1)^T \omega).$$

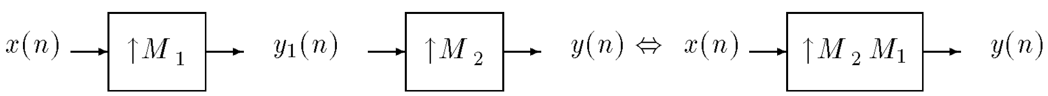

Figure 5: Upsampler-Upsampler (UU) Identity

5.2 Downsampler-Downsampler

A cascade of downsamplers implies a product of the corresponding downsampling factors as shown in Fig. 6. Indeed from Eqn. 9 one has $[\downarrow M_2]\,[\downarrow M_1] = [\downarrow M_1 M_2]$ since

$$y(n) = y_1(M_2 n) = x(M_1 M_2 n).$$

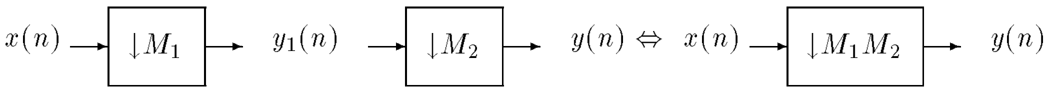

Figure 6: Downsampler-Downsampler (DD) Identity

5.3 Upsampler-Downsampler

In Fig. 7, for simplicity let $\mathcal{L}(M_2) \subseteq \mathcal{L}(M_1)$. Since upsampling onto $\mathcal{L}(M_1)$ is followed by by downsampling onto a sublattice of $\mathcal{L}(M_1)$, intuitively one expects this to be equivalent to upsampling on a *reduced* lattice. Indeed even more is true. Since $y(n) = [\downarrow M_2]\, y_1(n) = y_1(M_2 n)$, and $y_1(n) = [\uparrow M_1]\, x(n)$, from Eqn. 8

$$y(n) = \begin{cases} x(M_1^{-1} M_2 n) & \text{for } M_2 n \in \mathcal{L}(M_1) \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

If M is a gcld of M_1 and M_2 (i.e., $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ are sublattices of $\mathcal{L}(M)$), then $M_1 = MK_1$ and $M_2 = MK_2$. Therefore $M_1^{-1} M_2 = K_1^{-1} K_2$ and $M_2 n \in \mathcal{L}(M_1) \Leftrightarrow K_2 n \in \mathcal{L}(K_1)$ (from the nonsingularity of M). Moreover,

$$y(n) = \begin{cases} x(K_1^{-1} K_2 n) & \text{for } K_2 n \in \mathcal{L}(K_1) \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

From Eqn. 23 and Eqn. 24 we get $[\downarrow M_2]\,[\uparrow M_1] = [\downarrow K_2]\,[\uparrow K_1]$ implying the Upsampler-Downsampler identity in Fig. 7. In an upsampler-downsampler cascade one may always assume that M_1 and M_2 are left coprime. In particular, when $M_2 = M_1 K$, M_1 is indeed a gcld and therefore one can collapse it into upsampling by K as our intuition suggests!

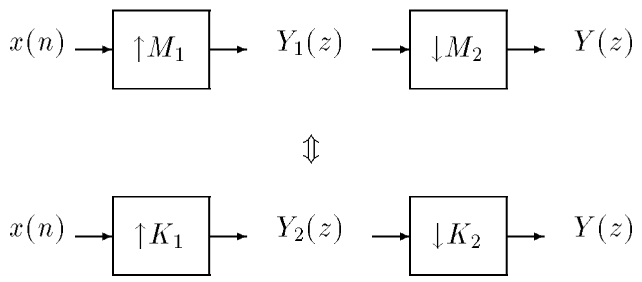

Figure 7: Upsampler-Downsampler (UD) Identity

5.4 Downsampler-Upsampler

In a downsampler-upsampler cascade there is no simplification. Assume downsampling by M_1 followed by upsampling by M_2. Then

$$y(n) = \begin{cases} x(M_1 M_2^{-1} n) & \text{for } n \in \mathcal{L}(M_2) \\ 0 & \text{otherwise.} \end{cases} .$$

If $M_1 = K_1 M$ and $M_2 = K_2 M$ (notice that there are no sublattice conditions), then $M_1 M_2^{-1} = K_1 K_2^{-1}$. However, $n \in \mathcal{L}(M_2)$ is *different* from $n \in \mathcal{L}(K_2)$ implying reduction is not possible *unless* M_2 and K_2 are related by a unimodular matrix (i.e., M_1 and M_2 are right coprime). This impossiblity of of reduction is even seen in the one dimensional case. It has to do the inherent irreversibility of the downsampling.

In summary, pure cascades of upsamplers/downsamplers can be reduced to a single upsampler/downsampler that is the product of the constituents. Upsamplers followed by downsamplers can always be reduced to the case of a single upsampler followed by a downsampler with left coprime matrices. No simplifications are possible where we have a downsampler followed by an upsampler.

5.5 Swapping Upsamplers and Downsamplers

Swapping of upsamplers and downsamplers is useful in many applications such as analyzing and designing *rational sampling rate filter banks* (Section 7).

Theorem 2 (The Swapping Theorem) *Given M_1, M_2 left coprime, there exist N_1, N_2 right coprime (and conversely) such that*

$$[\downarrow M_2]\,[\uparrow M_1] = [\uparrow N_1]\,[\downarrow N_2]\,. \tag{25}$$

Proof: The Aryabhatta/Bezout identity implies the existence of N_1 and N_2 such that

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{26}$$

In particular $N_1^T M_2^T = N_2^T M_1^T$ and $N_1^T P_2^T + N_2^T P_1^T = I$. Consider upsampling by M_1 followed by downsampling by M_2. From Eqn. 14)

$$\begin{aligned} Y(\omega) &= \frac{1}{|M_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} X(M_1^T M_2^{-T}(\omega - 2\pi k)) \\ &= \frac{1}{|M_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} N_1^T k) \\ &= \frac{1}{|N_2^T|} \sum_{k \in \mathcal{R}(M_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} N_1^T k) \quad \text{from Corollary 1} \\ &= \frac{1}{|N_2^T|} \sum_{k \in \mathcal{R}(N_2^T)} X(N_2^{-T} N_1^T \omega - 2\pi N_2^{-T} k) \qquad \text{from Lemma 1.} \end{aligned}$$

The last expression corresponds to downsampling by N_2 followed by upsampling by N_1. □

Theorem 3 (The Commuting Theorem) *Let M_1, M_2 be coprime, commuting matrices. Then*

$$[\uparrow M_1][\downarrow M_2] = [\downarrow M_1][\uparrow M_2] \quad \textit{and} \quad [\uparrow M_2][\downarrow M_1] = [\downarrow M_2][\uparrow M_1].$$

Proof: In the Swapping Theorem proof if we can choose $N_1 = M_1$ and $N_2 = M_2$ it suffices. Since $M_2 N_1 = M_1 N_2$ and $M_2 M_1 = M_1 M_2$, from Corollary 2, M_1 and M_2 are also right coprime. Therefore $M_2 M_1$ and $M_2 N_1$ are lcrms of M_2. Hence there exists unimodular U such that $M_2 N_1 U = M_2 M_1 = M_1 M_2 = M_1 N_2 U$. Therefore from Eqn. 26

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U^{-1}P_1 & U^{-1}P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} N_2 U & Q_2 \\ N_1 U & -Q_1 \end{bmatrix}. \end{aligned}$$

Relabeling UP_1 and UP_2 by P_1 and P_2, and from the fact that $N_1 U = M_1$ and $N_2 U = M_2$,

$$\begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} \begin{bmatrix} M_2 & Q_2 \\ M_1 & -Q_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{27}$$

From the above Aryabhatta/Bezout identity, and Theorem 2, we get Theorem 3. □

6 Commuting Filters and Upsamplers/Downsamplers

If we can analyze cascades of filters and up/downsamplers a complete set of tools will be in place. Filter-upsampler and downsampler-filter cascades are easily analyzed duals (these results are known as the *Noble* identities [18, 15]). Filter-downsampler and upsampler-filter cascades involve important concepts that play a major role in filter bank theory. This section introduces the new concept, *generalized* polyphase representations, giving more general results than are available in the literature. An important new identity we obtain is the Upsampler-Delay-Downsampler identity.

6.1 Filter-Upsampler

The Filter-Upsampler identity in Fig. 8 follows from the fact that

$$Y(z) = H(z^M)X(z^M) = H(z^M)\left\{[\uparrow M]\, X(z)\right\}.$$

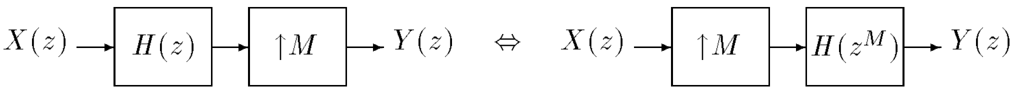

Figure 8: Filter-Upsampler (FU) Identity

6.2 Downsampler-Filter

The Downsampler-Filter identity in Fig. 9 follows from

$$\begin{aligned} y(n) &= \sum_{k\in \mathbf{Z}^d} h(k)x(Mn - Mk) = \sum_{k} h(M^{-1}Mk)x(Mn - Mk) \\ &= \sum_{k'\in \mathcal{L}(M)} h(M^{-1}k')x(Mn - k') = [\downarrow M]\left\{[\uparrow M]\, h(n) * x(n)\right\}. \end{aligned}$$

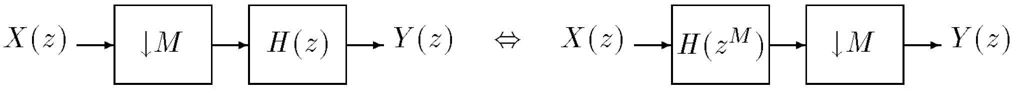

Figure 9: Downsampler-Filter (DF) Identity

6.3 Filter-Downsampler

The Filter-Downsampler identity is useful from both theoretical and computational points of view. Since downsampling throws away samples it is more efficient to downsample a signal *before* filtering. This leads to the *polyphase* representation [15] (also known as *lifting* in control theory and in mathematics [1]). The idea is to *"lift"* a scalar-valued sequence/function into a vector-valued sequence/function by blocking. The components of this *lifted* signal are referred to as *polyphase* components of the original sequence.

Let $\mathcal{S}(M)$ be a generalized set of representatives of $\mathcal{L}(M)$. The polyphase representation of $x(n)$ with respect to $\mathcal{S}(M)$ is the vector signal $x_p(n)$ whose components are given by

$$x_k(n) = x(Mn - k) = [\downarrow M]\{x(n-k)\} \quad \text{for } k \in \mathcal{S}(M). \tag{28}$$

There are $|M|$ polyphase components. In terms of $\mathcal{Z}$-transforms

$$X(z) = \sum_{k \in \mathcal{S}(M)} z^k X_k(z^M). \tag{29}$$

If $\mathcal{S}(M)$ is a generalized set of representatives so is $-\mathcal{S}(M)$. The polyphase representation of $x(n)$ with respect to $-\mathcal{S}(M)$ is called the *dual* polyphase representation of $x(n)$ with respect to $\mathcal{S}(M)$.

$$x_k(n) = x(Mn + k) = [\downarrow M]\{x(n+k)\} \quad \text{for } k \in \mathcal{S}(M).$$

If $\mathcal{S}(M) = \mathcal{R}(M)$, it is called the *first-orthant* (*synthesis* [20], or Type 1 [19]) polyphase representaion. The *dual first-orthant* polyphase representation is sometimes referred to as *analysis* or Type 2 polyphase representation.

There are infinitely many choices of generalized polyphase representation (since there are infinitely many generalized representatives of $\mathcal{L}(M)$). Eqn. 29 implies the generalized polyphase-inverse-polyphase (GPIP) and generalized inverse-polyphase-polyphase (GIPP) identities depicted in Fig. 10 and Fig. 11 respectively.

Let $X_p(z)$ and $H_p(z)$ denote the polyphase and dual polyphase representations of $X(z)$ and $H(z)$ respectively relative to $\mathcal{S}(M)$ (see Fig. 12). Then

$$\begin{aligned} Y(z) &= [\downarrow M]\left\{\sum_{k \in \mathcal{S}(M)} z^{-k} H_k(z^M)\right\}\left\{\sum_{l \in \mathcal{S}(M)} z^l X_l(z^M)\right\} \\ &= \left\{[\downarrow M] \sum_{k,l \in \mathcal{S}(M)} z^{l-k}\right\} H_k(z) X_l(z) \end{aligned}$$

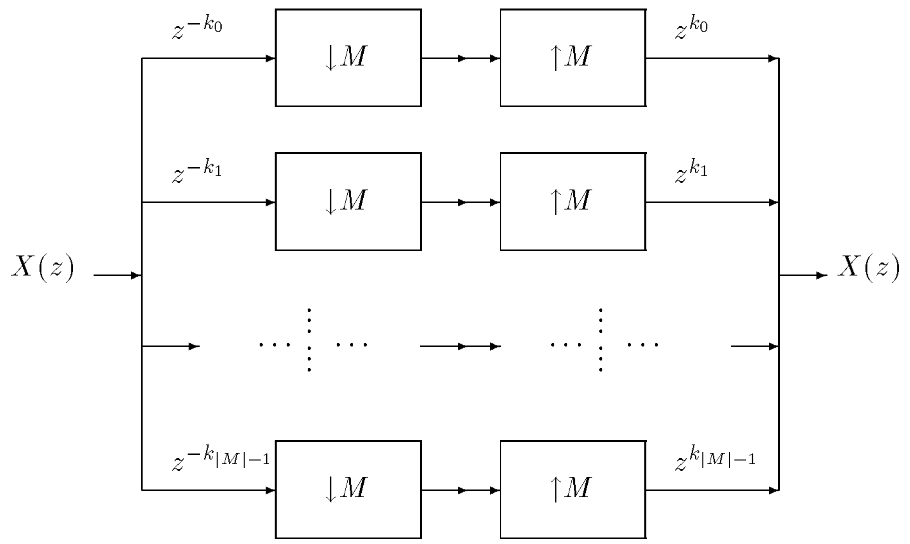

Figure 10: Generalized Polyphase-Inverse-Polyphase (GPIP) Identity

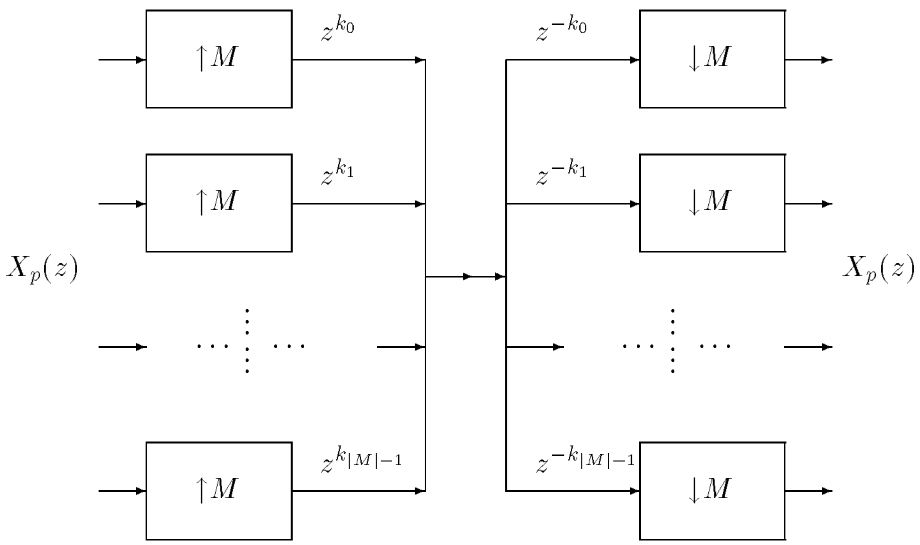

Figure 11: Generalized Inverse-Polyphase-Polyphase (GIPP) Identity

$$
\begin{aligned}
&= \sum_{k \in \mathcal{S}(M)} H_k(z) X_k(z) \quad \text{from Lemma 6} \\
&= H_p^T(z) X_p(z) \qquad (30)
\end{aligned}
$$

Remark: The analysis would have been complicated if $X(z)$ and $H(z)$ had not been represented

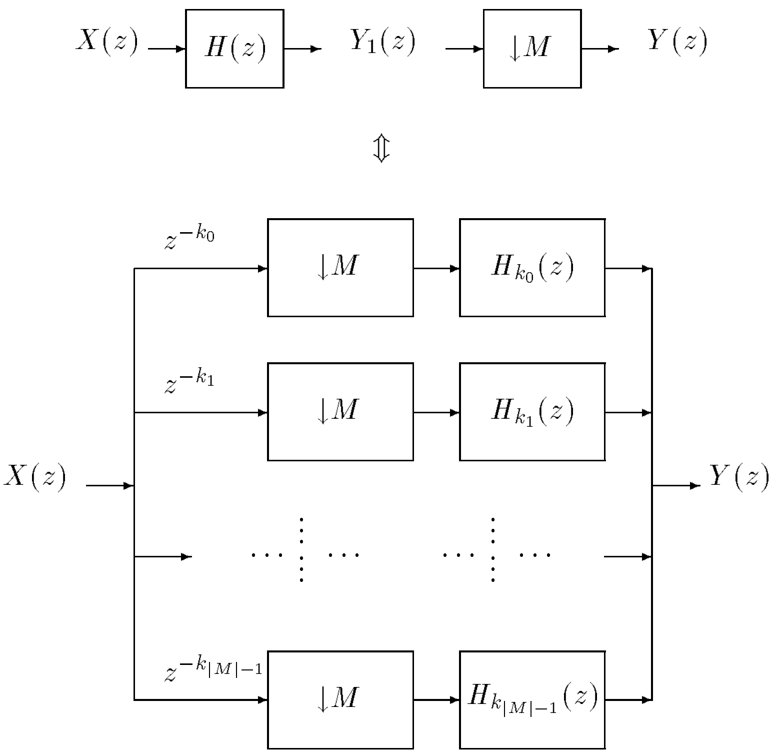

Figure 12: Filter-Downsampler (FD) Identity

in mutually dual polyphase representations. The literature usually deals only with first-orthant polyphase representations [20].

6.4 Upsampler-Filter

The filter-downsampler identity is simple if the *same* polyphase representation is used for the signals and the filter. Let $X_k(z)$, $Y_k(z)$ and $H_k(z)$ denote polyphase components of the signals and filter relative to $\mathcal{S}(M)$. Since $Y(z) = H(z)X(z^M) = \sum_{k \in \mathcal{S}(M)} z^k H_k(z^M) X(z^M) = \sum_{k \in \mathcal{S}(M)} z^k Y_k(z^M)$,

$$
Y_k(z) = H_k(z) X(z) \quad \text{for all } k \in \mathcal{S}(M). \qquad (31)
$$

Hence the mapping from $X(z)$ to $Y(z)$ may be represented as shown in Fig. 13.

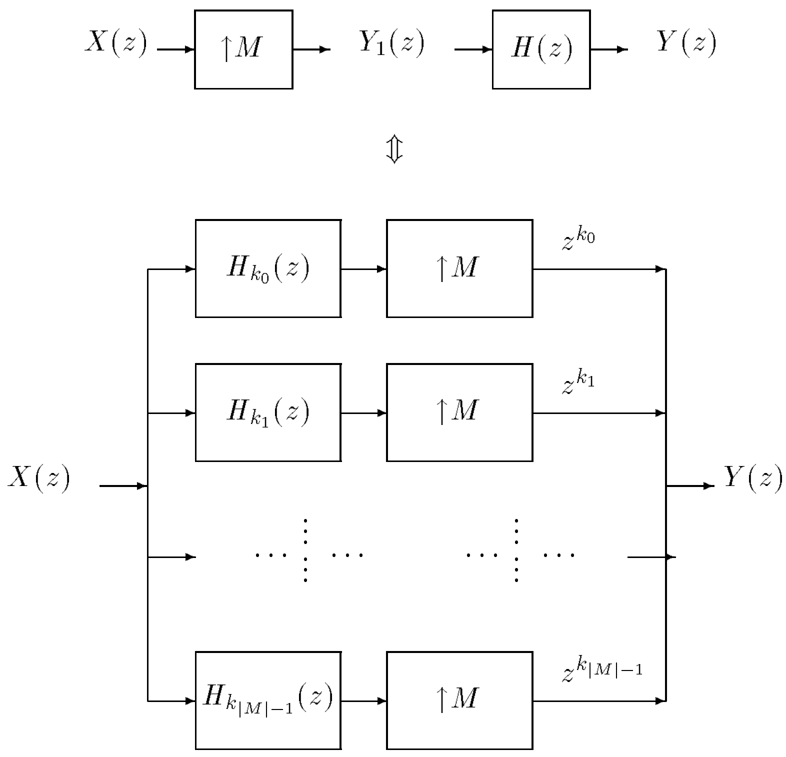

Figure 13: Upsampler-Filter (UF) Identity

6.5 Upsampler-Delay-Downsampler

One could use the UF or FD identities to shift the filter out from between the up/downsamplers. However, a look at Fig. 13 and Fig. 12 shows that this *will* introduce new delays between the upsampler and downsampler. So delays *squeezed* between up/downsamplers cannot be *released* by using the UF and FD identities. However, delays between down/upsamplers can be released by using DF or FU identitities.

Lemma 8 *Delays squeezed between an M_1-fold upsampler and M_2-fold downsampler can be released iff M_1 and M_2 are left coprime.*

Proof: Let $R = \mathrm{gcrd}(M_1, M_2)$. Then (Aryabhatta/Bezout identity) there exist matrices Q_2 and Q_1 such that $M_1Q_2+M_2Q_1 = R$. Any point in in $\mathcal{L}$ of the form Rm can be written as $M_1Q_2m+M_2Q_1m$. Therefore by using the FU and DF identities on the delay z^{Rm}, it can be released. If M_1, M_2 are left coprime one can take $R = I$ and the result follows. The converse follows similarly. Fig. 15 gives the steps involved in the reduction. □

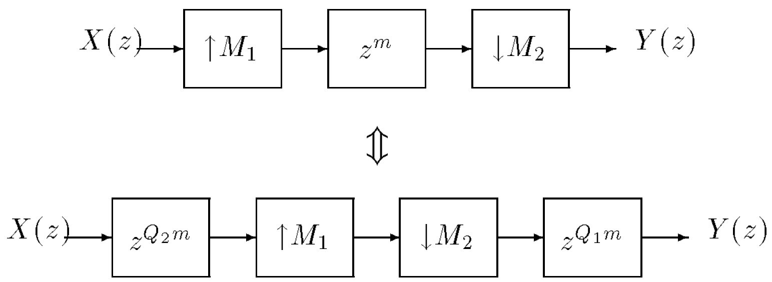

Figure 14: Upsampler-Delay-Downsampler (UΔD) Identity

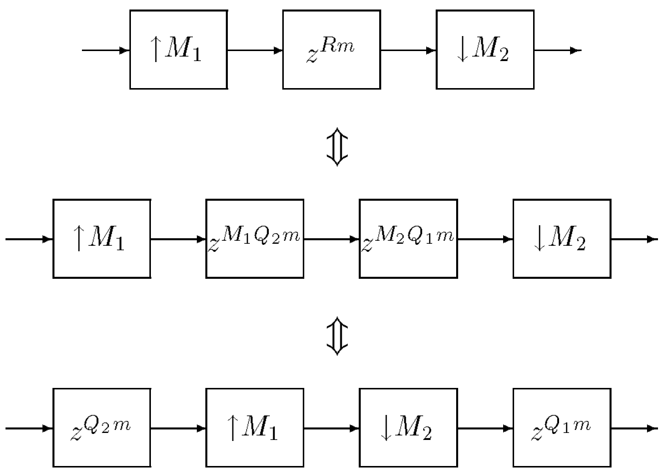

Figure 15: Upsampler-Delay-Downsampler Reduction Steps

7 Application to Rational Sampling Rate Filter Banks

Consider the rational sampling rate filter bank problem (Fig. 1) where the signal is spectrally split into nonuniform bins. The sampling rate change is different in each branch of the analysis bank. In one dimension this problem has been studied recently [12]. Using the tools developed in this paper, an algebraic reduction of the multidimensional rational sampling rate filter bank problem to a multidimensional uniform sampling rate filter bank problem is given.

7.1 Multidimensional Rational Sampling FilterBanks

In Fig. 1 let P_i and Q_i be left coprime such that the "sampling-feasibility" condition, $\sum_i |P_i| \,/\, |Q_i| = 1$, holds. This condition is not sufficient for the existence of perfect reconstruction rational sampling rate filter banks even in one dimension [12]. In any case, we algebraically reduce the problem to a $|Q|$-channel filter bank problem for some appropriate Q. The result is a direct generalization of the one dimensional result in [12]. The reduction is algebraic and says nothing about the relationship between the responses of the original and reduced filters. The reduction is a two step procedure, one to slide the upsampler in each branch past the filter and the downsampler and the other to then replace the downsampler by the least common left multiple of the downsamplers (that are different from the original downsamplers due to the first step).

7.2 Rational Filter Bank Reduction: Transform 1

Along any branch in Fig. 1 if the matrices corresponding to the upsampler and downsampler are left coprime one can transform the structure. For example, consider the cascade: $[\uparrow M_1]$, filter, $[\downarrow M_2]$ with M_1, M_2 left coprime. This can be converted to the following cascade: filter, delay, $[\downarrow N_1]$, $[\uparrow N_2]$ and delay with N_1, N_2 right coprime and $M_1 N_2 = M_2 N_1$. An outline of this reduction is as follows (see Fig. 16): (1) Use the UF identity to swap $[\uparrow M_1]$ and $H(z)$, (2) Use the UΔD identity (Lemma 8) to slide delays introduced in Step 1 along each polyphase branch and (3) Use the Swapping Theorem (Theorem 2) to introduce $[\downarrow N_1]$ and $[\uparrow N_2]$ along each branch.

The delays $z^{Q_2 k_i}$ in Fig. 16 can be absorbed into the filters H_{k_i}. Moreover the delays $z^{Q_1 k_i}$ can be thought of as implementing a generalized polyphase representation of $Y(z)$ with respect to N_2, the upsampler. This fact is important since the original problem reduces to the problem of design of the modified filters in Fig. 17 with only downsamplers. So see this notice the following

Aryabhatta/Bezout identity:

$$\begin{bmatrix} N_2 & Q_2 \\ N_1 & -Q_1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ M_1 & -M_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since N_1 and Q_1 are left coprime, from Theorem 1 (Eqn. 16b) $Q_1\mathcal{R}(M_1) \bmod N_1 = \mathcal{R}(N_1)$ showing that the generalized representatives in question is $\mathcal{S}(N_1) = Q_1\mathcal{R}(M_1)$.

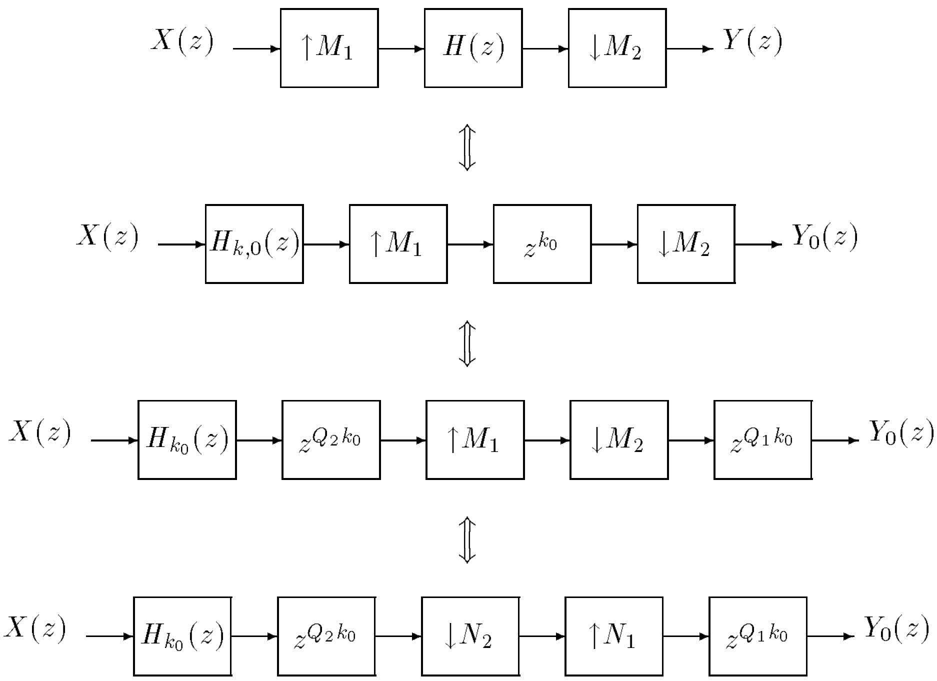

Figure 16: Rational Filter Bank Reduction: Steps in Transform 1

7.3 Rational FB Reduction: Transform 2

After Transform 1 the downsampler in each branch of the original filter bank are not the same. By making these dowsamplers the same the reduction to a uniform sampling rate filter bank would be complete. The idea is to replace $[\downarrow N]$ by $[\downarrow NL]$, for some L, followed by an inverse polyphase transform corresponding to L as shown in Fig. 19. The steps in tranform 2 are as follows: (1) Use GPIP identity with respect to L as in Fig. 10 and (2) Use DF identity to slide the delays after the downsampler to before it. This derivation of Transform 2 is much simpler than (even) the 1-d counterpart in [12].

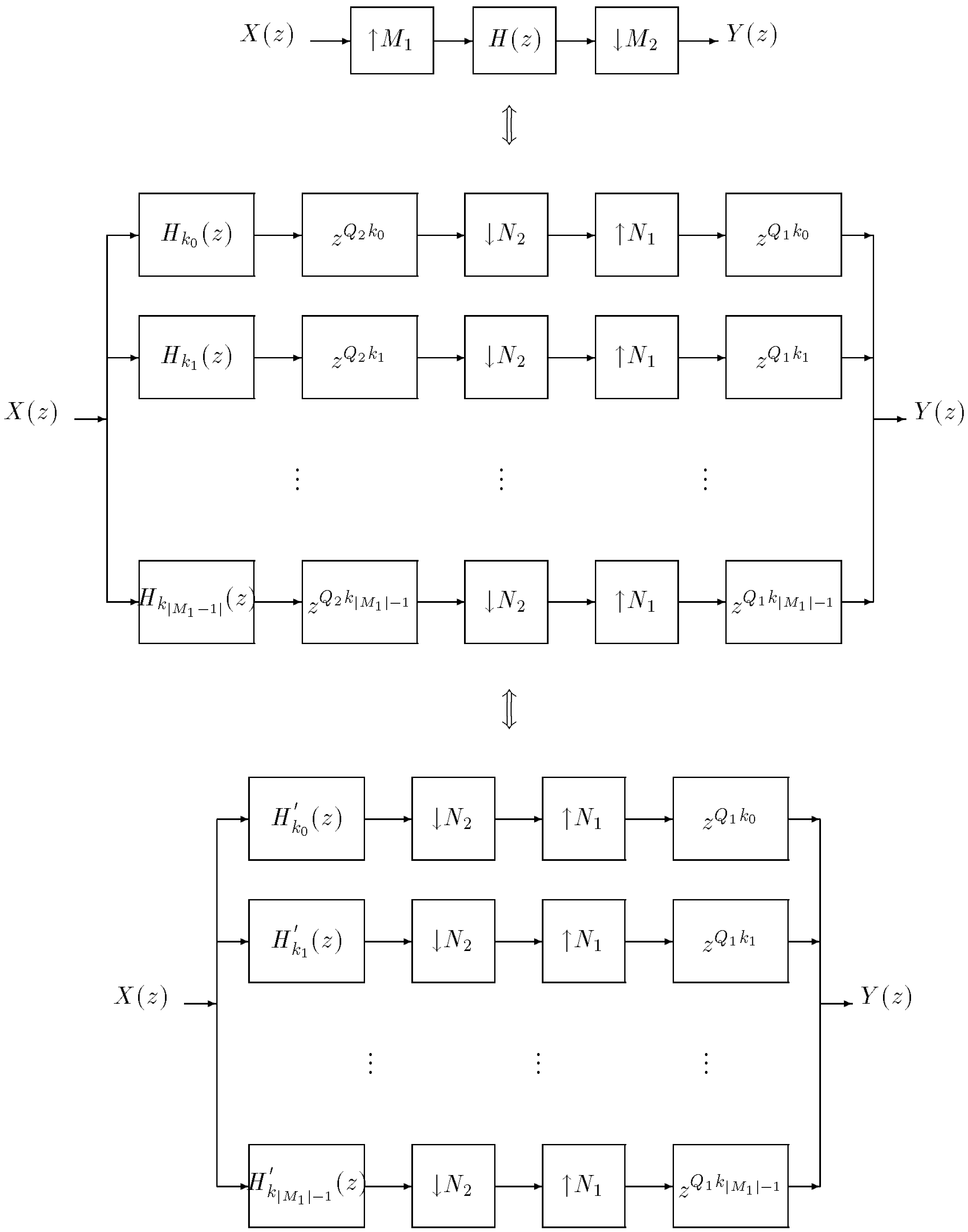

Figure 17: Rational Filter Bank Reduction: Transform 1

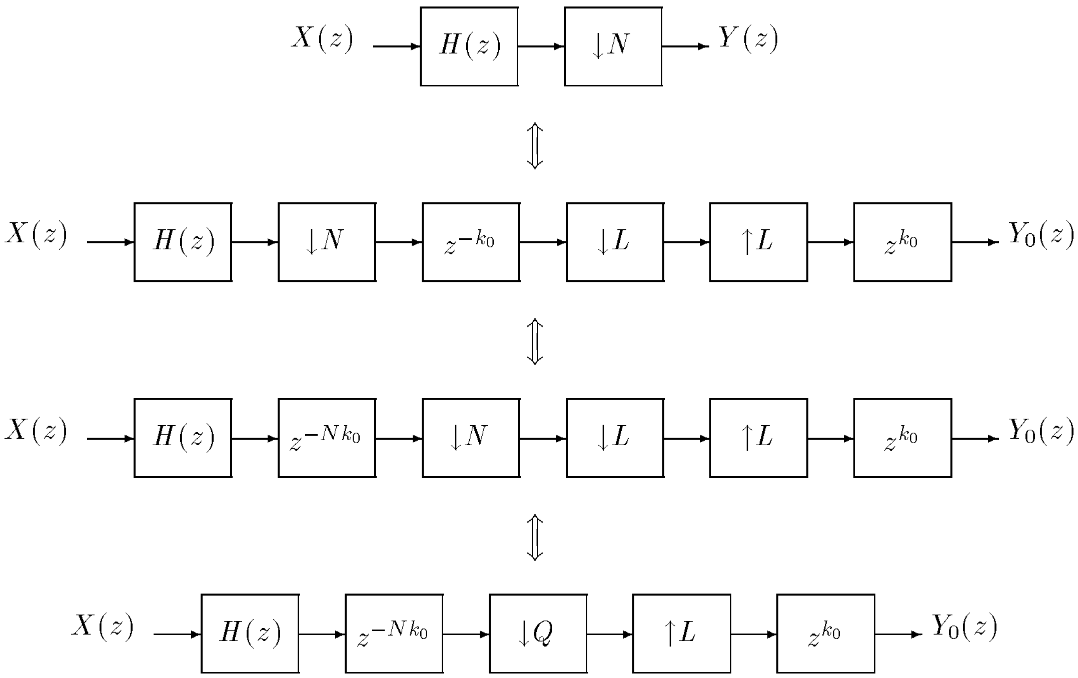

Figure 18: Rational FB Reduction: Steps in Transform 2

7.4 The Reduction to a Uniform Filter Bank

Consider the original problem in Fig. 1. Transform 1 is applied in each branch. This expands the i^{th} branch into $|P_i|$ branches giving an analysis bank with $\sum_i |P_i|$ branches. The downsampler in each new branch is, say $\tilde{Q}_i$ (which is different from Q_i). Let Q be an lcrm of these matrices with $Q = \tilde{Q}_i L_i$ for suitable matrices L_i. Apply Transform 2 in each branch. The branch corresponding to $\tilde{Q}_i$ expands into $|L_i|$ branches giving a (uniform) analysis filter bank with $\sum_{i=0}^{N-1} |P_i| \, |L_i|$ branches. Since $|L_i| = |Q| / \left|\tilde{Q}_i\right|$ and (from Corollary 1) $\left|\tilde{Q}_i\right| = |P_i|$, $\sum_{i=0}^{N-1} |P_i| \, |L_i| = |Q|$. In summary one has a Q channel uniform band filter bank corresponding to the downsampling matrix Q. A final question is whether Transform 2 can be modified so that the generalized inverse polyphase transform blocks in Fig. 20 can be replaced by an ordinary **IPT** block. The answer is in the affirmative and is an easy consequence of the GPIP identity.

8 Generalization of Kovacevic's Theorem

The swapping theorem may also be applied to give a generalization of Kovacevic's Theorem on commuting upsamplers and downsamplers. Kovacevic' proof was in two dimensions [11] and involves

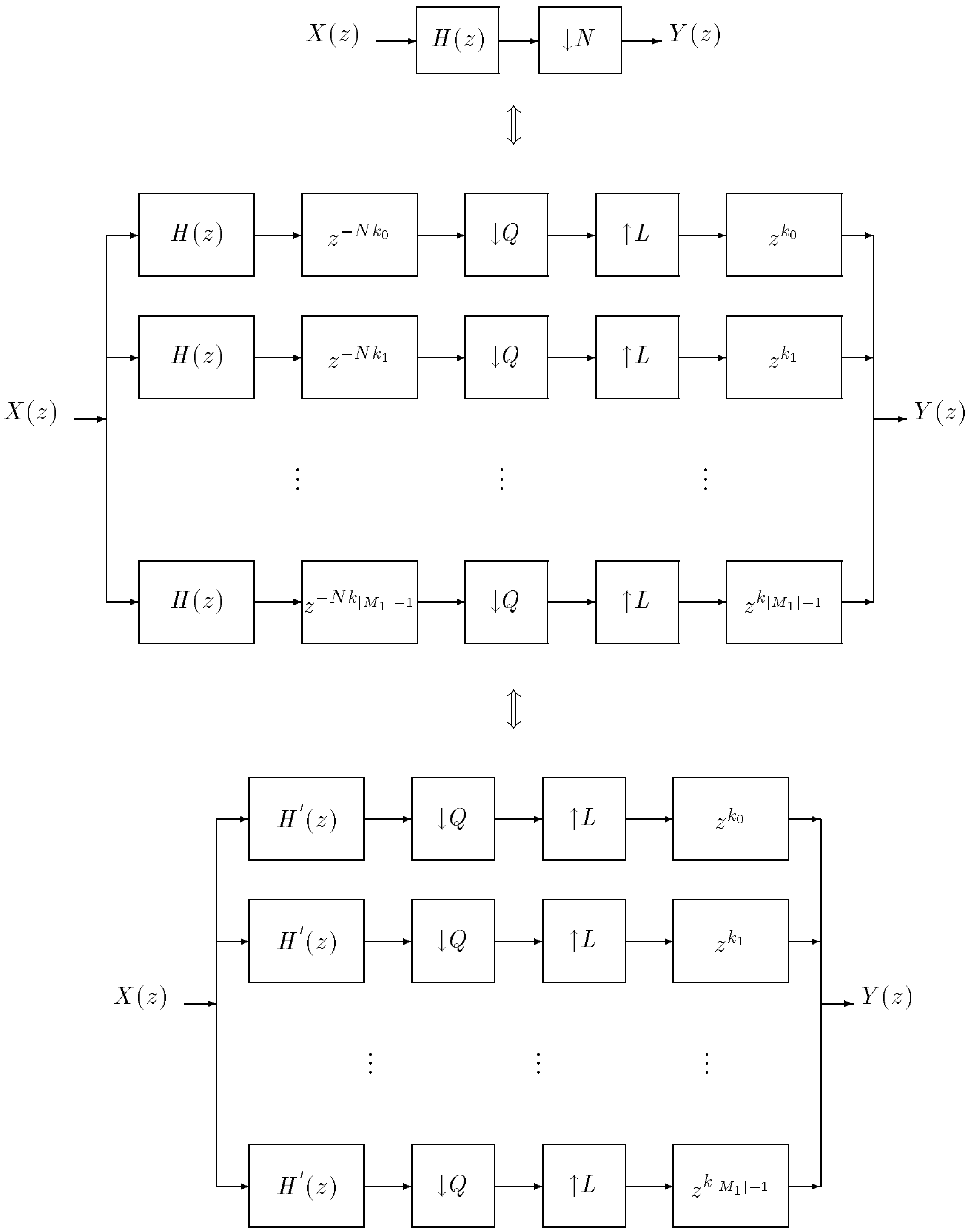

Figure 19: Rational FB Reduction: Transform 2

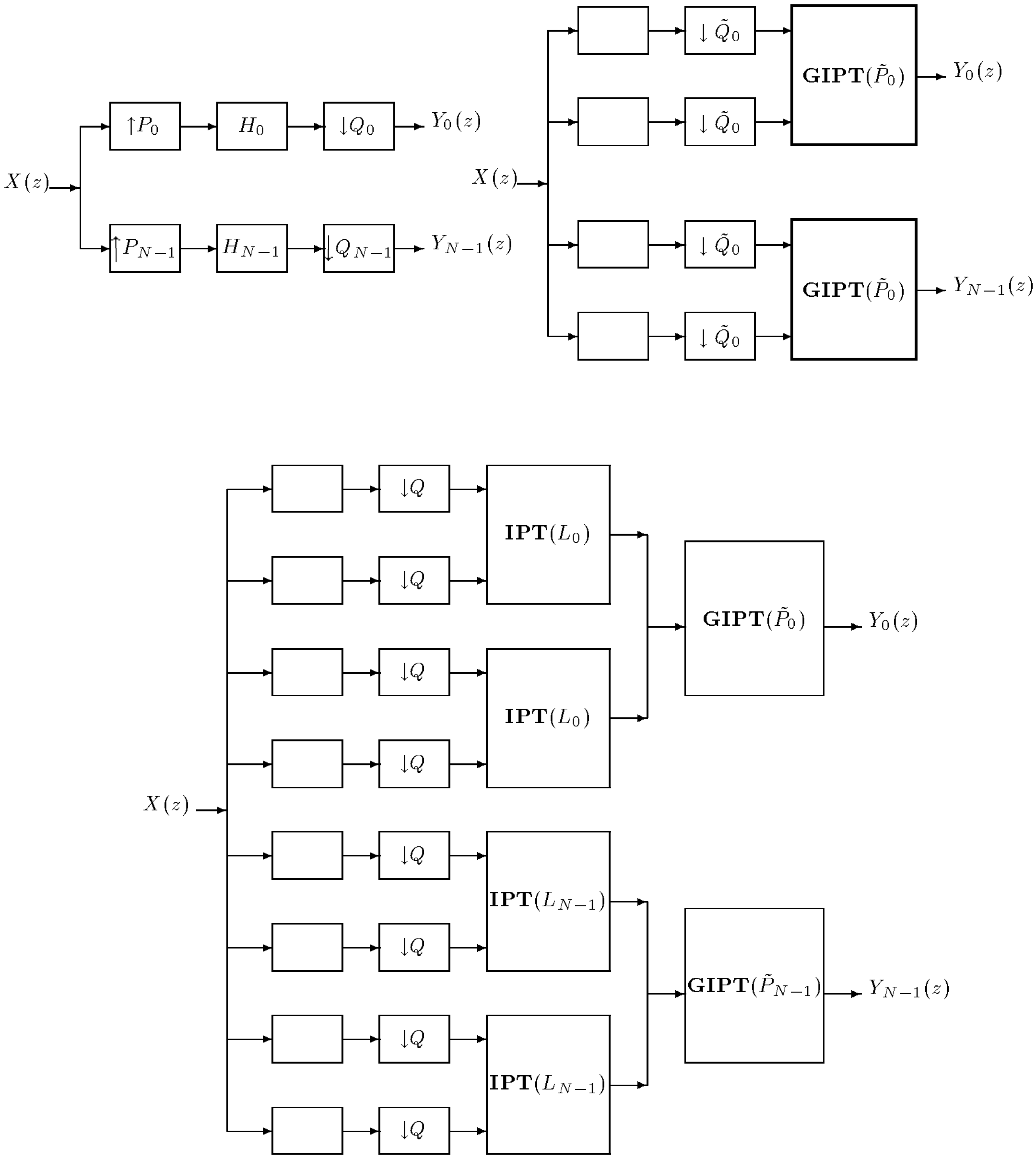

Figure 20: Reduction to Uniform Filter Bank

the detailed analysis of scalar lcms in a number of special situations and uses a special representation of integer matrices generating a given lattice. The power of the methods developed in this paper should be apparent from the simple proof of its generalization.

Theorem 4 (Kovacevic' Theorem) *Let* $M_1 M_2 = M_2 M_1$. *Then* $[\downarrow M_2]\,[\uparrow M_1] = [\uparrow M_1]\,[\downarrow M_2]$, *iff the determinant of generator of the the greatest common sublattice (gcs) of* $\mathcal{L}(M_1)$, *and* $\mathcal{L}(M_2)$, *viz.,* $\mathcal{L}(M_1) \cap \mathcal{L}(M_2)$, *is equal to* $\pm |M_1|\,|M_2|$.

Proof: Let M be a generator of the gcs of $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$. Then from Lemma. 1, there exist right coprime integer matrices K_1 and K_2 such that $M = M_1 K_1 = M_2 K_2$. M is an lcrm of M_1 and M_2. By assumption $\widehat{M} = M_1 M_2 = M_2 M_1$ and hence $\widehat{M}$ is a *common right multiple* of M_1 and M_2. This implies the existence of a matrix R (which is a gcrd of M_1 and M_2) such that $\widehat{M} = MR$. R is unimodular iff $|M| = \left|\widehat{M}\right| = |M_1|\,|M_2|$ (i.e., iff M_1 and M_2 are right coprime). By Corollary. 3 we can interchange upsampling and downsampling. $\square$

Remark: In the theorem in [11], the hypothesis is that M_1^T and M_2^{-T} commute, but this can be elegantly stated as "M_1 and M_2 commute".

The various multirate identities developed in this paper is tabulated in Table. 1.

Table 1: Summary of Multirate Identities

Filter-Filter (FF)	$H_1(z)$	$H_2(z)$	$H_1(z)$	$H_2(z)$
Downsampler-Downsampler (DD)	$[\downarrow M_1]$	$[\downarrow M_2]$	$[\downarrow M_1 M_2]$	
Upsampler-Upsampler (UU)	$[\uparrow M_1]$	$[\uparrow M_2]$	$[\uparrow M_2 M_1]$	
Upsampler-Downsampler (UD)	$[\uparrow M_1]$	$[\downarrow M_2]$	$[\uparrow K_1]$	$[\downarrow K_2]$
Downsampler-Upsampler (DU)	$[\downarrow M_1]$	$[\uparrow M_2]$	–	
Filter-Upsampler (FU)	$H(z)$	$[\uparrow M]$	$[\uparrow M]$	$H(z^M)$
Downsampler-Filter (DF)	$[\downarrow M]$	$H(z)$	$H(z^M)$	$[\downarrow M]$
Filter-Downsampler (FD)	$H(z)$	$[\downarrow M]$	$[\downarrow M]$	$H_k(z)$
Upsampler-Filter (UF)	$[\uparrow M]$	$H(z)$	$H_k(z)$	$[\uparrow M]$

9 Conclusion

In general in multiple dimensions, the factors in a cascade of upsamplers (or downsamplers) cannot be interchanged because of the non-commutativity of matrix multiplication. An upsampler followed by a downsampler can always be reduced to lowest terms. The reverse ordering cannot be so reduced (even in the scalar case). This is due to the fact that the downsampler is a irreversible operation.

The algebraic results are used to extend results on rational sampling rate filter banks to multiple dimensions. A number of interesting questions arise from this investigation. Can the non-uniqueness of the Aryabhatta/Bezout identity used in order to give simple classes of upsamplers/downsampers? Can this non-uniqueness at each stage of the rational to integer filter bank reduction be used to some advantage? These questions are currently under investigation.

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