Wavelet Interpolation and Approximate Solutions of Elliptic Partial Differential Equations

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Abstract

The paper formulates and proves a second order interpolation result for square-integrable functions by means of locally finite series of Daubechies' wavelets. Sample values of a sufficiently smooth function can be used as coefficients of a wavelet expansion at a fine scale, and the corresponding wavelet interpolation function converges in Sobolev norms of first order to the original function. This has applications to wavelet-Galerkin numerical solutions of elliptic partial differential equations.

1 Introduction

In 1988 Daubechies introduced in [5] a class of compactly supported wavelet functions which give a very nice decomposition of $L^2(\mathbf{R})$, and by tensor products, $L^2(\mathbf{R}^n)$. These have been studied by numerous authors and there have been a number of applications of these wavelet systems in engineering, technology as well as more abstract parts of mathematics and mathematical physics. For each positive integer N there is a family of compactly supported wavelets of dimension $N-1$ (see [13,14,19]). Out of this family one can choose wavelets which have specific properties for an application at hand. In her original paper Daubechies chose a particular wavelet system for each $N \geq 1$, which is distinguished by having the wavelet function having maximal vanishing moments, and by choosing a specfic choice of roots of certain polynomials which arise in the choice of the coefficients which define the wavelet system. These still seem to be most useful for many applications, especially in numerical approximation to solutions of differential equations, as they have important differentiability properties.

We will concentrate in this paper on this distinguished family of wavelets, which are defined by specific coefficient systems $\{a_0, a_1, \ldots, a_{2N-1}\}$, which define the scaling functions which we denote simply as $\phi(x)$ and which is denoted by ${}_N\phi(x)$ in [5]. The scaling function $\phi(x)$ satisfies the 2-scale difference equation

$$\phi(x) = \sum_{k=0}^{2N-1} a_k \phi(2x - k) \tag{1}$$

where the coefficients $\{a_0, a_1, \ldots, a_{2N-1}\}$ satisfy

$$\begin{aligned} \sum_{k=0}^{2N-1} a_k &= 2 \\ \sum_{k=0}^{2N-1} a_k a_{k+2l} &= 2\delta_{0,l}, \quad l \in \mathbf{Z} \end{aligned} \tag{2}$$

As mentioned above we are going to consider the specific choice of coefficients $\{a_n\}$ made by Daubechies for each N, and we will simply let ϕ denote this scaling function with the parameter N suppressed and understood. It is an important parameter as it determines the support of the scaling function (and associated wavelet functions), and this parameter plays a role in the calculations in this paper, as we shall see.

In Section 2 we review the well-known theory of the associated wavelet functions for the scaling functions $\phi(x)$ discussed above, their translations and rescalings and the corresponding orthonormal decompostion of $L^2(\mathbf{R})$ and $L^2(\mathbf{R}^2)$. In particular we recall the Mallat algorithm for determining wavelet series coefficients from one level to another, which is very useful in applications (see Mallat [11]), and plays an important role in our interpolation scheme and its applications in [8] and [20].

The Mallat algorithm leads to a natural interpolation by finite level wavelet series for a given function. In Section 3 we give a precise description of such a wavelet series interpolation, and with estimates which show that this interpolation approximates functions in $L^2(\mathbf{R}^2)$ and in the Sobolev space $H^1(\mathbf{R}^2)$ with convergence rates of the order of 2^{-2j} for the first case and 2^{-j} for the second case, where j is the level of the scaling functions involved in the interpolation series. The interpolation series can be expressed in different forms, but one of the simplest is given by:

$$f^j(x,y) = \sum_{p,q} f_{p,q}\phi_p^j(x)\phi_q^j(y), \tag{3}$$

where $\phi_p^j(x) := 2^{j/2}\phi(2^j x - p)$, for $p \in \mathbf{Z}$, is the standard rescaling of the given scaling function $\phi(x)$. The problem is: given a particular function f, what is the best choice of coefficients in (3) so that the series $f^j(x,y)$ best approximates the function f. There are many different versions of the word "best," and we present a specific choice which has a good convergence rate, and which is very useful in our applications. We demonstrate the convergence properties of this choice (which involves evaluating the function f at specific points when f is smooth), and apply it to proving estimates for wavelet-Galerkin solutions to elliptic differential equations in Section 4, which was the original motivation for these considerations. Our results are related to similar types of approximation results where the Fourier coefficient formulas of the form

$$c_k^j = \int f(x)\phi_k^j(x)dx$$

are approximated by evaluating f at specific points ([2,9,17,7]).

In related papers [20,8] we use this interpolation result in the context of specific Galerkin solutions of elliptic differential equation boundary value problems. In these papers we use a wavelet representation of the boundary data, the line integrals involved in the weak form of the differential equation, of the boundary itself, as well as the more traditional use (in the spirit of finite element methodology) for the solutions on the interior of the domain. The wavelet system provides a unifying common method of representing all of the elements of a boundary value problem (see [16]). The interpolation scheme presented in this paper is a key ingredient in going between "values of the wavelet expansion coefficients" and "values of the corresponding functions" which the wavelet series represent. The algorithms in [20,8] concentrate on the manipulation of the wavelet coefficients to find an approximation of the solution of the differential equation, just as in image processing the manipulation of the wavelet expansion coefficients plays the major role (see Mallat [11] and Zettler et al [21]).

2 Wavelet Series and the Mallat Algorithm

Let ϕ be the Daubechies scaling function of order N as discussed in Section 1, and let $\{a_0, \ldots, \alpha_{2N-1}\}$ be the corresponding coefficients for the 2-scale difference equation (1). The following summary contains results originally obtained in [5,11] (see also the books [12,3,15]).

The vector $(a_0, \ldots, a_{2N-1})$ is called the *scaling vector* for the system, and we define the associated *wavelet vector* $(b_0, \ldots, b_{2N-1})$ by defining

$$b_k := (-1)^{k+1} a_{2N-1-k} \tag{4}$$

This is the convention used in [10] and differs slightly from that used originally in [5]. We define the corresponding wavelet function $\psi(x)$ by

$$\psi(x) := \sum_{k=0}^{2N-1} b_k \phi(2x-k). \tag{5}$$

With this convention, both the scaling function $\phi(x)$ and the corresponding wavelet function $\psi(x)$ have the same compact support $[0, 2N-1]$. Moreover, we have the following properties of moments of these functions.

$$\begin{aligned} \int \phi(x)\, dx &= 1, \\ \int x\phi(x)\, dx &= \frac{1}{2} \sum_{k=0}^{2N-1} k a_k, \\ \int x^m \psi(x)\, dx &= 0, \quad m = 0, \ldots, N-1. \end{aligned} \tag{6}$$

Let us denote by

$$c := \frac{1}{2} \sum_{k=0}^{2N-1} k a_k, \tag{7}$$

the first moment of the function $\phi(x)$, as this will play a role in our calculations later. Moreover, ϕ and ψ are both differentiable of a certain finite order $\alpha(N)$, where $\alpha = O(N)$, with $\alpha(2) \cong 0.55$, $\alpha(3) \cong 1.088$, etc. (see [5], [6], [18]), where the latter two references give asymptotic formulas for $\alpha(n)$.

For $j, k \in \mathbf{Z}$, let us define

$$\phi_k^j(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_k^j(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbf{R},$$

and as usual we define $\phi_k(x) := \phi_k^0(x)$. The following properties of these functions are valid, as is easy to verify:

$$\begin{aligned} \operatorname{supp}(\phi_k^j) = \operatorname{supp}(\psi_k^j) &= \left[\frac{k}{2^j}, \frac{k+2N-1}{2^j}\right] && (8) \\ ||\phi_k^j||_{L^2(\mathbf{R})} &= 1 && (9) \\ \sum_{k\in\mathbf{Z}} \phi_k^j(x) &= 2^{j/2} \quad x \in \mathbf{R}. && (10) \end{aligned}$$

Property (10) is the partition of unity property (especially for $j = 0$). Moreover, the wavelet system

$$\{\phi_k(x), \psi_k^j(x), \quad k \in \mathbf{Z}, j \in \mathbf{Z}^+\} \tag{11}$$

is an orthonormal basis for $L^2(\mathbf{R})$, where we denote by $\mathbf{Z}^+$ the nonnegative integers (we denote the positive integers by $\mathbf{N}$).

Let, for $j \in \mathbf{Z}$,

$$V_j := \text{closure}(\text{span}\{\phi_k^j : k \in \mathbf{Z}\}). \tag{12}$$

Then $L^2(\mathbf{R}) = \bigcup_j V_j$, that is, for any function $f \in L^2(\mathbf{R})$, if we let P_j denote the orthogonal projection $L^2(\mathbf{R}) \to V^j$, then $P_j f$ converges to f in the L^2 norm. We recall that the coefficients of $P_j f$ are given by the classical formulas

$$P_j f(x) = \sum_{k\in\mathbf{Z}} f_k^{j,0} \phi_k^j(x), \tag{13}$$

where

$$f_k^{j,0} = \int_{\mathbf{R}} f(x)\phi_k^j(x)\, dx, \tag{14}$$

since the ϕ_k^j's are an orthonormal basis for V_j. We will denote coefficients of the scaled scaling functions generically by $f_*^{*,0}$ and of the scaled wavelet functions by $f_*^{*,1}$, letting the superscripts $\{0\}$ and $\{1\}$ distinguish between the two cases.

If we let W_j denote the orthogonal complement of V_j in V_{j+1}, then we see that, for a fixed $J \in \mathbf{Z}^+$,

$$V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_{J-1} = V_J. \tag{15}$$

Moreover, if we let

$$f(x) = \sum_{k\in\mathbf{Z}} f_k^{0,0}\phi_k^0(x) + \sum_{j\in\mathbf{Z}^+}\sum_{k\in\mathbf{Z}} f_k^{j,1}\psi_k^j(x),$$

for $f \in L^2(\mathbf{R})$, then we see that we have from (15),

$$\sum_{k\in\mathbf{Z}} f_k^{J,0}\phi_k^J(x) = \sum_{k\in\mathbf{Z}} f_k^{0,0}\phi_k^0(x) + \sum_{j=0}^{J-1}\sum_{k\in\mathbf{Z}} f_k^{j,1}\psi_k^j(x). \tag{16}$$

In this case all of the coefficients are given by integral formulas of the type:

$$f_k^{j,0} = \int_{\mathbf{R}} f(x)\phi_k^j(x)\,dx, \quad j = 0,\dots,J-1, \tag{17}$$

$$f_k^{j,1} = \int_{\mathbf{R}} f(x)\psi_k^j(x)\,dx, \quad j = 0,\dots,J-1. \tag{18}$$

Moreover, it follows from the 2-scale difference equation that one can determine the coefficients on the right hand side of (16) in terms of the coefficients on the left hand side, and conversely. This is the Mallat algorithm [11], and the formulas will be given below. First we want to remark that if we consider *any* expansion of the form

$$\tilde{f}^J(x) := \sum_{k\in\mathbf{Z}} f_k^{J,0}\phi_k^J(x), \tag{19}$$

where the coefficients $f_k^{J,0}$ are prescribed in some fashion (perhaps relating to some given L^2 function f or not), then because of (15), we see that there is a corresponding expansion of the form

$$\tilde{f}^J(x) = \sum_{k\in\mathbf{Z}} f_k^{0,0}\phi_k^0(x) + \sum_{j=0}^{J-1}\sum_{k\in\mathbf{Z}} f_k^{j,1}\psi_k^j(x), \tag{20}$$

and the coefficients in (20) are determined in terms of the coefficients in (19) and the converse is true. This is the Mallat algorithm in general. It doesn't have anything to do with the coefficients being generalized Fourier coefficients of some given function as in (17) and (18).

The Mallat algorithm is given as follows. Suppose we are given an expansion of the form (19), then we want to determine the coefficients of the corresponding lower order expansion of the type (20). By hypothesis we have that $\tilde{f}^J \in V_J$. By (15) we see that

$$\tilde{f}^j(x) = \sum_{k\in\mathbf{Z}} f_k^{0,0}\phi_k(x) + \sum_{j=0}^{j-1}\sum_{k\in\mathbf{Z}} f_k^{j,1}\psi_k^j(x) \tag{21}$$

for some coefficients $\{f_k^{0,0}, f_k^{j,1}\}$. By multiplying both sides of (21) by $\phi_k^j(x)$ and integrating , and using (1) and the orthonormality of the wavelet functions, we find that, for a fixed j and k,

$$\begin{aligned}
f_k^{j,0} &= \sum_l f_l^{j+1}\int \phi_l^{j+1}(x)\phi_k^j(x)\,dx \\
&= \frac{1}{\sqrt{2}}\sum_{l,n} f_l^{j+1}a_n\int \phi_l^{j+1}(x)\phi_{2k+n}^{j+1}(x)\,dx \\
&= \frac{1}{\sqrt{2}}\sum_{l,n} f_l^{j+1}a_n\delta_{2k+n}^l \\
&= \frac{1}{\sqrt{2}}\sum_{l} f_l^{j+1}a_{l-2k},
\end{aligned} \tag{22}$$

and similarly by (5),

$$f_k^{j,1} = \frac{1}{\sqrt{2}}\sum_l f_l^{j+1}b_{l-2k}. \tag{23}$$

We note that we assume in these calculations that $a_i = 0$ for $i < 0$ or $i > 2N-1$.

Conversely, by knowing the expansion (20), we can determine the coefficients in (19) recursively by the intermediate expansions in terms of scaling functions at order less than J of the form

$$\tilde{f}^j(x) = \sum_k f_k^{j,0}\phi_k^j(x)$$

where, for $j = 0,\ldots,J$,

$$\begin{aligned}
f_k^{j,0} &= \sum_l (f_l^{j-1,0}\int \phi_l^{j-1}(x)\phi_k^j(x)\,dx + f_l^{j-1,0}\int \psi_l^{j-1}(x)\phi_k^j(x)\,dx) \\
&= \frac{1}{\sqrt{2}}\sum_l (f_l^{j-1,0}a_{k-2l} + f_l^{j-1,0}b_{k-2l}).
\end{aligned}$$

In the next section we will, for a given input function of a certain type, formulate a specfic expansion of the form (19) which will form a good approximation in the $L^2(\mathbf{R})$ norm to the given function. This expansion will be related to the lower order expansion of the form (20) by the Mallat algorithm. In general the coefficients in this expansion will not be given by Fourier coefficient formulas of the type (18) and (18), but will correspond instead to *evaluating the function* f at a well chosen lattice of points.

3 Wavelet Interpolation

In this section we prove the following interpolation theorem. The theorem is stated in $\mathbf{R^2}$ for simplicity, but is true in $\mathbf{R^n}$ as well, as is apparent from the proof. Recall that c is a constant depending on the wavelet system defined by (7) .

Theorem 3.1 *Assume the function* $f \in C^2(\bar{\Omega})$*, where* Ω *is a bounded open set in* $\mathbf{R^2}$*. Let, for* $j \in \mathbf{N}$*,*

$$f^j(x,y) := \frac{1}{2^j} \sum_{p,q \in \Lambda} f\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right) \phi_p^j(x) \phi_q^j(y), \quad x,y \in \Omega, \tag{24}$$

where the index set $\Lambda = \{i \in Z : \operatorname{supp}(\phi_i^j) \cap \Omega \neq \emptyset\}$*. Then*

$$||f - f^j||_{L^2(\Omega)} \leq C(1/2^j)^2, \tag{25}$$

and

$$||f - f^j||_{H^1(\Omega)} \leq C/2^j,$$

where C *is a constant depending only on the diameter of* Ω*,* N*, and the maximum modulus of the first and second order derivatives of* f *on* $\bar{\Omega}$*.*

Proof: Without loss of generality, we can assume that Ω is a square with edge length L. Then the number of p's (and q's) in the summation in (24) is no more than $2^j L + (2N-1)$.

We first show that

$$||f - f^j||_{L^2(\Omega)} \leq C/2^j. \tag{26}$$

For each $p, q \in \Lambda$, using the Taylor expansion for f at a point $(x,y) \in \Omega$, we have

$$f\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right) = f(x,y) + \frac{\partial f}{\partial x}(\xi_p, \xi_q)\left(\frac{p+c}{2^j} - x\right) + \frac{\partial f}{\partial y}(\xi_p, \xi_q)\left(\frac{q+c}{2^j} - y\right), \tag{27}$$

for some (ξ_p, ξ_q) on the segment connecting $\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right)$ and (x,y).

Substituting (27) into (24) and taking (10) into account, one derives

$$\begin{aligned} 2^j(f - f^j)(x,y) &= \sum_{p,q\in\Lambda} \frac{\partial f}{\partial x}(\xi_p, \xi_q)\left(\frac{p+c}{2^j} - x\right)\phi_p^j(x)\phi_q^j(y) + \\ &\quad + \sum_{p,q\in\Lambda} \frac{\partial f}{\partial y}(\xi_p, \xi_q)\left(\frac{q+c}{2^j} - y\right)\phi_p^j(x)\phi_q^j(y). \end{aligned} \tag{28}$$

Squaring both sides of (28),

$$\begin{aligned} 2^{2j}|f - f^j|^2(x,y) &\leq C2^j\left(\sum_{p\in\Lambda}\left|\phi_p^j\left(\frac{p+c}{2^j} - x\right)\right|\right)^2 + \left(\sum_{q\in\Lambda}\left|\phi_q^j\left(\frac{q+c}{2^j} - y\right)\right|\right)^2 \\ &= C2^j\left(\sum_{p,p'\in\Lambda}\left|\phi_p^j\phi_{p'}^j\right|\left|\left(\frac{p+c}{2^j} - x\right)\left(\frac{p'+c}{2^j} - x\right)\right| + \right. \\ &\quad + \sum_{q,q'\in\Lambda}\left|\phi_q^j\phi_{q'}^j\right|\left|\left(\frac{q+c}{2^j} - y\right)\left(\frac{q'+c}{2^j} - y\right)\right|, \end{aligned} \tag{29}$$

where the constant C depends only on the maximum modulus of the first derivatives of f on $\bar{\Omega}$. Therefore, by (9) and Hölder's inequality, (29) gives

$$\begin{aligned} 2^{2j}||f - f^j||^2_{L^2(\Omega)} &\leq \frac{C2^j}{2^{2j}}\left\{\sum_{p\in\Lambda}\sum_{|p'-p|\leq 2N-2}\int|\phi_p^j\phi_{p'}^j|\,dx + \right. \\ &\quad \left. + \sum_{q\in\Lambda}\sum_{|q'-p|\leq 2N-2}\int|\phi_q^j\phi_{q'}^j|\,dy\right\} \\ &\leq C, \end{aligned} \tag{30}$$

where the constant C depends on the constant in (29), N and the diameter of the domain Ω. Hence (26) follows from (30).

We now show that

$$||f - f^j||_{L^2(\Omega)} \leq C(1/2^j)^2, \tag{31}$$

and

$$||\nabla f - \nabla f^j||_{L^2(\Omega)} \leq C/2^j. \tag{32}$$

For each $p, q \in \Lambda$, using the Taylor expansion for f at point $(x, y) \in \Omega$, we have

$$\begin{aligned} f\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right) &= f(x,y) + \frac{\partial f}{\partial x}(x,y)\left(\frac{p+c}{2^j} - x\right) + \frac{\partial f}{\partial y}(x,y)\left(\frac{q+c}{2^j} - y\right) + \\ &\quad + \tfrac{1}{2}R(x,y), \end{aligned} \tag{33}$$

where

$$\begin{aligned} R(x,y) &= \frac{\partial^2 f}{\partial x^2}(\xi_p, \xi_q)\left(\frac{p+c}{2^j} - x\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\xi_p, \xi_q)\left(\frac{p+c}{2^j} - x\right)\left(\frac{q+c}{2^j} - y\right) + \\ &\quad + \frac{\partial^2 f}{\partial y^2}(\xi_p, \xi_q)\left(\frac{q+c}{2^j} - y\right)^2 \end{aligned} \tag{34}$$

for some (ξ_p, ξ_q) on the segment connecting $\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right)$ and (x, y). Therefore,

$$\begin{aligned} 2^j(f^j - f)(x,y) &= \sum_{p,q\in\Lambda} \frac{\partial f}{\partial x}\left(\frac{p+c}{2^j} - x\right)\phi_p^j(x)\phi_q^j(y) + \\ &\quad + \sum_{p,q\in\Lambda} \frac{\partial f}{\partial y}\left(\frac{q+c}{2^j} - y\right)\phi_p^j(x)\phi_q^j(y) \\ &\quad + \sum_{p,q\in\Lambda} R(x,y)\phi_p^j(x)\phi_q^j(y). \end{aligned} \tag{35}$$

To analyze the first two terms on the right hand side of (35), let us consider the wavelet expansion of the function x over the admissible x-slice of the domain Ω in terms of the scaling functions. Write

$$x = \sum_{k\in\Lambda} d_k \phi_k^j(x),$$

for some constants d_k's. Then a simple calculation implies that for each k,

$$d_k = 2^{-3j/2}[k + c],$$

where c is the first moment of ϕ (7). Hence

$$\sum_p p\phi_p^j(x) = 2^{3j/2}x - 2^{j/2}c. \tag{36}$$

This implies that the first two summations in (35) vanish, and we obtain,

$$2^j(f^j - f)(x,y) = \sum_{p,q\in\Lambda} R(x,y)\phi_p^j(x)\phi_q^j(y). \tag{37}$$

But

$$|R(x,y)| \leq C\left\{\left|\frac{p+c}{2^j} - x\right|^2 + \left|\frac{p+c}{2^j} - x\right|\left|\frac{q+c}{2^j} - y\right| + \left|\frac{q+c}{2^j} - y\right|^2\right\}, \tag{38}$$

where the constant C depends only on the maximum of the second order derivatives of f. Therefore

$$\begin{aligned}
2^{2j}|f^j - f|^2 \quad &\leq \quad C\left\{\sum_{p,q\in\Lambda}\left(\left|\frac{p+c}{2^j} - x\right|^2 + \left|\frac{p+c}{2^j} - x\right|\left|\frac{q+c}{2^j} - y\right|\right.\right. \\
&\quad \left.\left. + \left|\tfrac{q+c}{2^j} - y\right|^2\right)\phi_p^j(x)\phi_q^j(y)\right\}^2 \\
&= \quad C\sum_{p,q\in\Lambda}\sum_{p',q'\in\Lambda}\left(\left|\frac{p+c}{2^j} - x\right|^2 + \left|\frac{p+c}{2^j} - x\right|\left|\frac{q+c}{2^j} - y\right| + \left|\frac{q+c}{2^j} - y\right|^2\right)\cdot \\
&\quad \cdot\left(\left|\frac{p'+c}{2^j} - x\right|^2 + \left|\frac{p'+c}{2^j} - x\right|\left|\frac{q'+c}{2^j} - y\right| + \left|\frac{q'+c}{2^j} - y\right|^2\right)\cdot \\
&\quad \cdot\phi_p^j(x)\phi_{p'}^j(x)\phi_q^j(y)\phi_{q'}^j(y).
\end{aligned} \tag{39}$$

Hence by (9),

$$2^{2j}||f - f^j||^2_{L^2(\Omega)} \leq C\sum_{p.q\in\Lambda}(1/2^j)^4 \leq C(1/2^{2j}),$$

and (31) is proved.

To prove estimate (32), we again use the Taylor expansion (33). From (24), we have

$$\frac{\partial f^j}{\partial x}(x,y) = \sum_{p,q\in\Lambda} f\left(\frac{p+c}{2^j}, \frac{q+c}{2^j}\right)\frac{\partial\phi_p^j(x)}{\partial x}\phi_q^j(y). \tag{40}$$

So (40) and (10) imply

$$
\begin{aligned}
2^j \frac{\partial f^j}{\partial x}(x,y) &= \sum_{p,q\in\Lambda} \frac{\partial f}{\partial x}\left(\frac{p+c}{2^j}-x\right)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y)+ \\
&\quad + \sum_{p,q\in\Lambda} \frac{\partial f}{\partial y}\left(\frac{q+c}{2^j}-y\right)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y) + \sum_{p,q\in\Lambda} R(x,y)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y).
\end{aligned}
\tag{41}
$$

Again by (10), one derives

$$
\begin{aligned}
2^j \frac{\partial f^j}{\partial x}(x,y) &= \sum_{p,q\in\Lambda} \frac{\partial f}{\partial x}\frac{p+c}{2^j}\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y) + \sum_{p,q\in\Lambda} R(x,y)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y) \\
&= \frac{1}{2^{j/2}}\frac{\partial f}{\partial x}\sum_{p,q\in\Lambda} p\frac{\partial \phi_p^j(x)}{\partial x} + \sum_{p,q\in\Lambda} R(x,y)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y).
\end{aligned}
\tag{42}
$$

Recalling (36), one has

$$
\begin{aligned}
\sum_{p\in\Lambda} p\frac{\partial \phi_p^j(x)}{\partial x} &= \frac{\partial}{\partial x}\left(\sum_{p\in\Lambda} p\phi_p^j(x)\right) \\
&= \frac{\partial}{\partial x}\left(2^{3j/2}x - c\sum_{p\in\Lambda}\phi_p^j(x)\right) = 2^{3j/2}.
\end{aligned}
\tag{43}
$$

Substituting (43) into (42), one has

$$
2^j\left(\frac{\partial f^j}{\partial x}-\frac{\partial f}{\partial x}\right)(x,y) = \sum_{p,q\in\Lambda} R(x,y)\frac{\partial \phi_p^j(x)}{\partial x}\phi_q^j(y). \tag{44}
$$

Therefore by (44) and (38),

$$
\begin{aligned}
2^{2j}\left|\frac{\partial f^j}{\partial x}-\frac{\partial f}{\partial x}\right|^2(x,y) &\le C\left\{\sum_{p,q\in\Lambda}\left(\left|\frac{p+c}{2^j}-x\right|^2+\left|\frac{p+c}{2^j}-x\right|\left|\frac{q+c}{2^j}-y\right|+\right.\right.\\
&\quad\left.\left.+\left|\frac{q+c}{2^j}-y\right|^2\right)\frac{\partial\phi_p^j(x)}{\partial x}\phi_q^j(y)\right\}^2\\
&= C\sum_{p,q\in\Lambda}\sum_{p',q'\in\Lambda}\left(\left|\frac{p+c}{2^j}-x\right|^2+\left|\frac{p+c}{2^j}-x\right|\left|\frac{q+c}{2^j}-y\right|+\right.\\
&\quad\left.+\left|\frac{q+c}{2^j}-y\right|^2\right)\left(\left|\frac{p'+c}{2^j}-x\right|^2+\left|\frac{p'+c}{2^j}-x\right|\left|\frac{q'+c}{2^j}-y\right|+\right.\\
&\quad\left.+\left|\frac{q'+c}{2^j}-y\right|^2\right)\frac{\partial\phi_p^j(x)}{\partial x}\frac{\partial\phi_{p'}^j(x)}{\partial x}\phi_q^j(y)\phi_{q'}^j(y).
\end{aligned}
\tag{45}
$$

Note that

$$
\int\left|\frac{\partial\phi_p^j(x)}{\partial x}\right|^2 dx\le 2^{2j}\int|\phi\prime|^2\,dx. \tag{46}
$$

Then combining (10) and (46), using the Hölder inequality, and again, as in (30), noticing the relation between p and p'; q and q', we obtain

$$
2^{2j}\left\|\frac{\partial f}{\partial x}-\frac{\partial f^j}{\partial x}\right\|^2_{L^2(\Omega)}\le C\left(\sum_{p,q\in\Lambda}\left(\frac{1}{2^j}\right)^4(2^{2j}2^{2j})^{1/2}\right)\le C, \tag{47}
$$

where the constant C depends on the constant in (40), N and the diameter of the domain Ω. That is,

$$
\left\|\frac{\partial f}{\partial x}-\frac{\partial f^j}{\partial x}\right\|_{L^2(\Omega)}\le C/2^j.
$$

Similarly, we can derive

$$
\left\|\frac{\partial f}{\partial y}-\frac{\partial f^j}{\partial y}\right\|_{L^2(\Omega)}\le C/2^j.
$$

Then (32) follows readily from above and the Theorem is proved. □

4 Estimates for Elliptic Partial Differential Equations

Using Theorem 3.1 as an interpolation one can easily derive some error estimates for the wavelet solutions of certain elliptic partial differential equations. In this section we derive one such estimate for the case of a Galerkin approximation to a solution of a specific boundary value problem as an example of this phenomenon. The result is easily generalized to other boundary value problems in higher dimensions provided that a general coercive system is considered. we recall that if W is a finite-dimensional subspace of some Sobolev space in which one looks for a weak solution of a differential equation, we look for a solution (the Galerkin solution) of the finite dimensional linear system of equations obtained from the weak form of the differential equation by restricting the test functions *and* the solution function to both be in the approximation space W (see [4] for a discussion of this).

More specifically let Ω be a bounded open set in $\mathbf{R}^2$ with a Lipschitz boundary. Let

$$W^j := \{f \in V^j : \mathrm{supp} f \cap \bar{\Omega} \neq \emptyset\}.$$

Since Ω is bounded it's clear that W^j is a finite-dimensional subspace of V^j. This will be our Galerkin approximation space. If we consider the Sobolev space $H^1(\Omega)$, then we have clearly

$$W^j \subset H^1(\Omega),$$

and thus we can use elements of W^j to represent approximations to a solution of a differential equation which are in $H^1(\Omega)$.

Consider the elliptic equation

$$-\Delta u + u = f, \quad \text{in } \Omega, \tag{48}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \tag{49}$$

where n is the unit outward normal vector of $\partial\Omega$. If u is a solution of (48) and (49) and if h is a test function in $H^1(\Omega)$, then multiplying (48) by h and integrating by parts over Ω, one has from (49) that

$$\int_\Omega \nabla u \nabla h dxdy + \int_\Omega uhdxdy = \int_\Omega fhdxdy + \int_{\partial\Omega} ghds. \tag{50}$$

Solving (48) and (49) is equivalent to finding $u \in H^1(\Omega)$ so that (50) is satisfied for all $h \in H^1(\Omega)$. Let u^j be a solution of (50) where $u^j \in W^j$ and (50) is satisfied for all $h \in W^j$ (the Galerkin approximation). Assume that the wavelet parameter $N \geq 2$.

Theorem 4.1 *If u is a solution to* (48) *and* (49)*, then*

$$||u - u^j||_{H^1(\Omega)} \leq \frac{C}{2^j},$$

where C depends on the diameter of Ω, on N, and the maximum modulus of the first and second derivatives of u.

Proof: We show first that if $v \in W^j$, then

$$||u - u^j||_{H^1(\Omega)} \leq ||u - v||_{H^1(\Omega)}. \tag{51}$$

To see this, we let, for $v \in V^j$, $w = u^j - v$. Then since u and u^j both satisfy (50) for any $h \in W^j$ and $w \in W^j$, one has

$$\int_\Omega (\nabla u - \nabla u^j) \cdot \nabla w \, dxdy + \int_\Omega (u - u^j) w \, dxdy = 0. \tag{52}$$

Therefore, by (52) and the Schwarz inequality,

$$\begin{aligned} ||u - u^j||^2_{H^1(\Omega)} &= \int_\Omega (\nabla u - \nabla u^j)^2 \, dxdy + \int_\Omega (u - u^j)^2 \, dxdy \\ &= \int_\Omega (\nabla u - \nabla u^j) \cdot (\nabla u - \nabla v - \nabla w) \, dxdy + \\ &\quad + \int_\Omega (u - u^j)(u - v - w) \, dxdy \\ &= \int_\Omega (\nabla u - \nabla u^j) \cdot (\nabla u - \nabla v) \, dxdy \\ &\quad + \int_\Omega (u - u^j)(u - v) \, dxdy \\ &\leq ||u - u^j||_{H^1(\Omega)} ||u - v||_{H^1(\Omega)}. \end{aligned} \tag{53}$$

If $u = u^j$, (51) is obviously true, otherwise, (51) also holds because of (53). Let now $v = \tilde{u}^j$ be defined to be the interpolation of u as given by Theorem 3.1, then Theorem 4.1 follows from (51) and Theorem 3.1. $\square$

We now generalize this last result by replacing the data f and g by perturbations of f and g before we make the Galerkin approximation. In practice, this is what one does, as it is then necessary to use numerical approximations for f and g in a given problem. Suppose that we are given f and g as in (48) and (49), where we suppose both f and g are C^2 functions in $\mathbf{R^2}$ which restrict to Ω and $\partial\Omega$ respectively, and let f^j and g^j be wavelet interpolations for f and g of order j as in Theorem 3.1. Let now u^j be the wavelet-Galerkin solution to

$$\int_\Omega \nabla u \nabla h dxdy + \int_\Omega uhdxdy = \int_\Omega f^j hdxdy + \int_{\partial\Omega} g^j hds, \tag{54}$$

and then we obtain

Corollary 4.2 *If u is a solution to* (48) *and* (49)*, and u^j is the wavelet-Galerkin solution to* (54)*, then*

$$||u - u^j|| \leq \frac{C}{2^j},$$

where C depends on the diameter of Ω, on N, and on the maximum modulus of the first derivatives of u.

Proof: From (50) and (52) we obtain

$$\int_\Omega (\nabla u - \nabla u^j) \cdot \nabla w \, dxdy + \int_\Omega (u - u^j) w \, dxdy = \int_\Omega (f - f^j) w \, dxdy + \int_{\partial\Omega} (g - g^j) w \, ds.$$

Therefore, similar to the derivation of (53), one has

$$\begin{aligned} ||u - u^j||^2_{H^1(\Omega)} \leq \; & ||u - u^j||_{H^1(\Omega)} ||u - v||_{H^1(\Omega)} + \textstyle\int_\Omega |(f - f^j)(u^j - v)| \, dxdy + \\ & + \textstyle\int_{\partial\Omega} |(g - g^j)(u^j - v)| \, ds. \end{aligned} \tag{55}$$

Note that if $\partial\Omega$ is Lipschitz, then one has the following well-known inequality [1].

$$\int_{\partial\Omega} |f| \, ds \leq C ||f||_{H^1(\Omega)}, \tag{56}$$

where C depends only on the dimension and Ω. Combining the above two inequalities (55) and (56), estimates for f, g from Theorem 3.1, replacing v by $\tilde{u}^j$, and using the Schwarz inequality again, the Corollary follows. $\square$

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