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Abstract

Recently multiplicity M, K-regular, orthonormal wavelet bases (that have implications in transform coding applications) have been constructed by several authors [1, 2, 3]. This paper describes and parameterizes the *cosine-modulated* class of multiplicity M wavelet tight frames (WTFs). In these WTFs (like in the general multiplicity 2 case in [4]), the scaling function uniquely determines the wavelets. This is in contrast to the general multiplicity M case, where one has to, for any given application, design the scaling function *and* the wavelets. Several design techniques for the design of K regular cosine-modulated WTFs are described and their relative merits discussed. Wavelets in K-regular WTFs may or may not be smooth. Since coding applications use WTFs with short length scaling and wavelet vectors (since long filters produce ringing artifacts which is undesirable in, say, image coding), many smooth designs of K regular WTFs of short lengths are presented. In some cases, analytical formulas for the scaling and wavelet vectors are also given. In many applications smoothness of the wavelets is more important than K regularity. We define smoothness of filter banks and WTFs using the concept of total variation and give several useful designs based on this smoothness criterion. Optimal design of cosine-modulated WTFs for signal representation is also described. All WTFs constructed in this paper are orthonormal bases, and we believe that this is always the case.

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1 Introduction

Recently orthonormal bases of compactly supported wavelets have received considerable attention in the signal processing community, both as a tool for signal analysis ([5, 6, 7, 8]) and signal representation ([9, 10, 11, 12, 13, 14]). Several authors have tried to use wavelets for image compression [10, 15]. It is well known that image coding using wavelets is a special case of subband coding using particular sets of filters called scaling and wavelet vectors. However, this specialization and the mathematical theory of wavelets that comes with it, gives new results and insights into the performance of image compression methods using subband coding [10].

Transform coding (using block transforms, subband coders or lapped transforms) is known to be an effective technique for the representation of speech and image signals at medium to low bit rates [16]. While subband coders solve the blocking effects associated with block transforms (like the DCT), they come with added cost in design and implementation. Subband coders using unitary filter banks (also known as lapped transforms) partially alleviate this problem of design and implementation. Moreover, they are precisely the class of filter banks associated with orthonormal wavelet bases. More recently, a new class of unitary filter banks called cosine-modulated filter banks (also known as extended lapped transforms) have been developed. These filter banks are easy to design and implement.

In her landmark paper [4], I. Daubechies constructs compactly supported orthonormal wavelet bases by constructing a special sequence, the scaling vector h_0 that satisfies a set of linear and quadratic conditions. The quadratic conditions ensure that h_0 is a filter in a two-channel unitary filter bank. The linear conditions (one of which is *necessary* for the existence of wavelets) ensure that the Fourier transform of h_0 is sufficiently flat at the origin. These linear conditions are also known as *regularity* conditions because for the wavelet to be $K-1$ times differentiable it is *necessary* that h_0 is K-regular (i.e., $K-1$ derivatives of $H_0(\omega)$ vanish at the origin). However, regularity of h_0 does not guarantee any degree of differentiability (or smoothness) of the wavelet. Another important consequence of regularity is that the zeroth through $K-1$ moments of the wavelet are zero.

Using the well-known theory of M-channel unitary filter banks, the authors and several others have recently constructed orthonormal bases (or tight frames) of multiplicity M, K-regular, compactly supported wavelets [1, 3, 2]. Multiplicity M wavelet bases give a more flexible time-frequency resolution tradeoff than octave-band multiplicity 2 wavelet bases [3] One difference between the multiplicity 2 case and the multiplicity M case is that in the former once the scaling vector and hence the scaling function is determined, the wavelet vector h_1, and the corresponding wavelet is precisely determined modulo translation. However, in the multiplicity M case, there are $\binom{M-1}{2}$ free parameters that determine the $M-1$ wavelet vectors h_i (and hence the corresponding wavelets), once the scaling vector is given.

Since there is a rich family of compactly supported WTFs, one can pick a tight frame that is suited to any particular application. This involves choosing the scaling function, and then the wavelets. In the wavelet literature there has been some recent work on the choice of "optimal" and "robust" scaling function or equivalently the scaling vector [12, 13, 11]. The scaling vector is generically of length $N = MK$, and determined by $(M-1)(K-1)$ parameters. All properties of the multiresolution analysis is determined by the scaling function. However, when $M > 2$, to get a WTF one has to design the wavelets too. For large M the design of the WTF becomes difficult because of the enormous number of free parameters ($\binom{M-1}{2}$).

In this paper we solve this problem by constructing a class of multiresolution analyses where the scaling function uniquely determines the wavelets. We obtain this by using the recently developed theory of unitary cosine-modulated filter banks [17, 18, 19, 20]. We explicitly parameterize all compactly supported multiplicity M cosine-modulated WTFs. There are a number of advantages to this approach

1. Firstly, the Fourier transform of the wavelets and scaling functions occupy adjacent bands in the spectrum, achieving the primary goal of flexible time-frequency resolution - the very reason for considering multiplicity M WTFs.

2. Secondly, the design is simple. While a general WTF is determined by $\binom{M-1}{2} + (M-1)(K-1)$ parameters, where $N = MK$, a cosine-modulated WTF is determined by just $N/4$ parameters. For large M or N, the savings is remarkable. For example, if one wants to design a multiplicity 64 WTF, the cosine-modulated approach seems to be the only viable one!

3. Analysis/synthesis computations in these wavelet bases can be done very efficiently using a combination of two-channel unitary filter banks and the DCT.

In many applications in numerical analysis and approximation theory using wavelets it is desirable to have first $K-1$ moments of the wavelets vanish [21, 22, 14]. This also means that the scaling function and its translates can approximate polynomials of degree $K-1$ exactly. In the multiplicity M case, one can explicitly give formulas for the autocorrelation of K regular scaling vectors [3]. Such an explicit characterization of K-regular cosine-modulated scaling vectors does not seem to be feasible. However, using the parameterization developed in this paper, one can design K regular scaling vectors and hence WTFs. In general one has to solve a set of non-linear equations, and hence K-regular WTFs are designed numerically. However, for small K and M, they may be designed analytically. Our analytical examples, illustrate the important fact that regularity of the scaling vector may not imply regularity of the wavelets in the WTF. Also for fixed N and K, there may be multiple solutions (that are *not distinct spectral factors* of a fixed autocorrelation function) to the non-linear equations. However, for small K and M we give smooth WTF designs using this approach. For moderate M and K, the length of the scaling vector grows rapidly, making the numerical design difficult. To cope with this problem, we introduce a novel technique, the moments minimization method, which produces smooth designs more consistently than the earlier approach.

In most signal processing applications the regularity of the scaling vector (or the moment vanishing property of the wavelets) is not important. However, the smoothness of the scaling function and wavelets (i.e., their being "visually" non-fractal) is important. For example, in image processing applications tree-structured hierarchical lapped transforms have been used for a while. If the tree is deep, then the lowpass filter iterates and the scaling function samples can be identified. Said differently, in such applications iterates of the scaling vector must not *look bad*. We propose two new techniques for the design of such "smooth" scaling vectors, without using the usual regularity approach. The first is based on the intuitive idea that if one were to design a scaling function that optimally represents an artificial smooth signal (like a sinusoid, for example), then the scaling function would tend to be smooth. Using the theory of design of optimal multiresolution analyses in [12], we show how smooth cosine-modulated WTFs can be designed. The second approach, based on a new definition of smoothness, seems to be well suited for designing smooth WTFs. This approach also is shown to lead to better filter bank designs.

All explicit examples of WTFs constructed in this paper are orthonormal bases. We conjecture that all cosine-modulated WTFs are orthonormal bases.

2 Overview of Relevant Results

2.1 Wavelet Frames and Filter Banks

Fig. 1) shows an M-channel filter bank with analysis filters h_i and synthesis filters g_i. The filter bank is a perfect reconstruction (PR) filter bank if $y(n) = x(n)$. Under certain conditions, PR filter

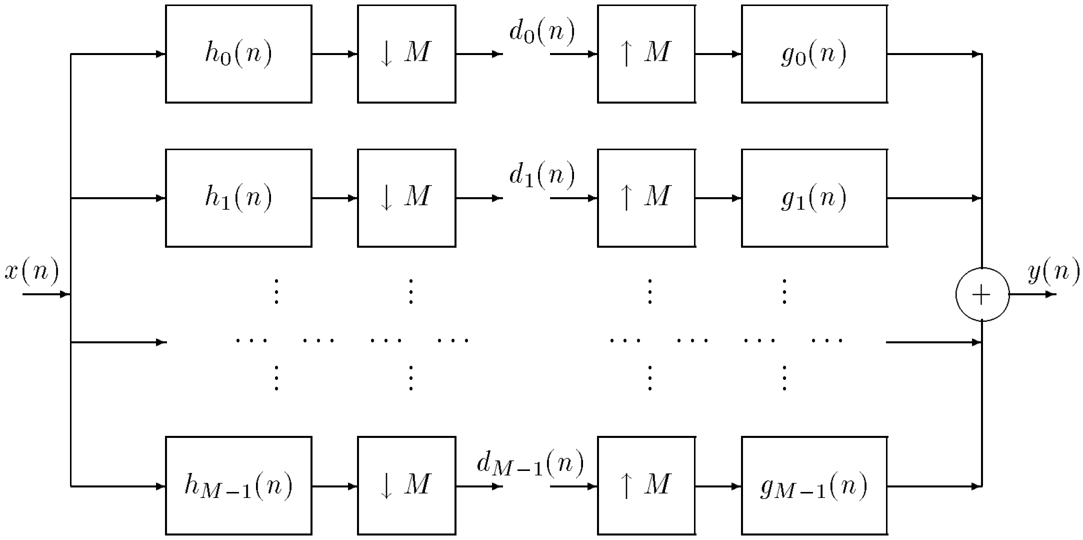

Figure 1: An M Channel Filter Bank

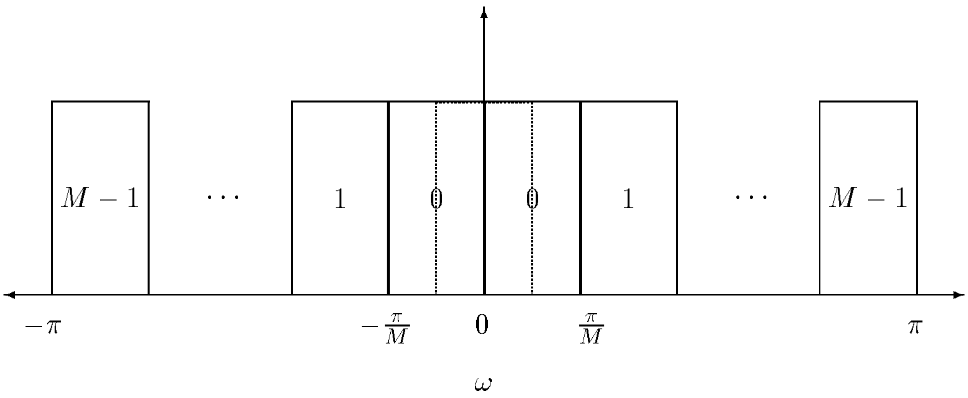

Figure 2: Ideal Responses in M Channel Filter Bank

banks are associated with wavelet frames for $L^2(\mathbb{R})$ [1, 23]. This association is a correspondence between the filters and the scaling and wavelet vectors associated with the wavelet frame (see Table 1). PR filter banks where the analysis and synthesis filters satisfy $g_i(n) = h_i(-n)$ are called

Table 1: Relationship between Filter Bank Theory and Wavelet Theory

Filter Bank Theory	Wavelet Theory
Lowpass Analysis filter h_0	Scaling Vector/Function (h_0, ψ_0)
Bandpass Analysis filter h_i	Scaling Vector/Function (h_i, ψ_i)
Lowpass Synthesis filter g_0	Dual Scaling Vector/Function $(g_0, \tilde{\psi}_0)$
Bandpass Synthesis filter g_i	Dual Scaling Vector/Function $(g_i, \tilde{\psi}_i)$

unitary filter banks. Unitariness of a filter bank is equivalent to the following quadratic conditions on the filters.

$$\sum_k h_i(k) h_j(k + Ml) = \delta(l)\delta(i - j) \tag{1}$$

The term unitary comes from the fact that the polyphase component matrix of the filter bank is unitary on the unit circle [24]. Unitary filter banks are also known as paraunitary filter banks [24].

A unitary filter bank where the lowpass filter satisfies the additional linear constraint,

$$\sum_{k=0}^{N-1} h_0(k) = \sqrt{M}, \tag{2}$$

gives rise to a wavelet tight frame [1]. This filter is the unitary scaling vector, and the remaining filters in the filter bank are the unitary wavelet vectors. The scaling and wavelet vectors determine the scaling function and wavelets defined by the following equations:

$$\psi_i(t) = \sqrt{M} \sum_k h_i(k) \psi_0(Mt - k) \qquad\qquad i \in \{0, \ldots, M - 1\} \tag{3}$$

The $(M - 1)$ wavelets, $\psi_i(t), i \in \{1, \ldots, M - 1\}$, their translates, and dilates by powers of M form a wavelet tight frame. That is, any $f(t) \in L^2(\mathbb{R})$ can be represented in the form

$$f(t) = \sum_{i=1}^{M-1} \sum_{j,k} \langle f, \psi_{i,j,k}(t) \rangle \, \psi_{i,j,k},$$

where $\psi_{i,j,k}(t) = M^{j/2} \psi_i(M^j t - k)$.

It is well known in the multiplicity 2 case that the smoothness (degree of differentiability) of the scaling function is usually hard to determine [4, 25, 26]. The situation is the same in the multiplicity M case also. However, a necessary condition for $K-1$ derivatives of $\psi_0(t)$ (and hence the wavelets) to exist is that the zeroth through $K-1$ moments of the wavelets vanish. This is equivalent to the Fourier transform of the scaling vector being K^{th} order flat at the origin. Scaling vectors with this property are called K-regular scaling vectors. More formally,

Definition 1 *a scaling vector is said to be K regular if its $\mathcal{Z}$-transform is of the form*

$$H_0(z) = (1 + z^{-1} + \cdots + z^{-(M-1)})^K P(z) \tag{4}$$

for maximal possible K, and $P(z)$ a polynomial in z^{-1}.

It can be shown that the minimal length K-regular scaling vectors are generically of length $N = MK$ [3]. For example, the smallest length scaling vector is the Haar scaling vector which is given by

$$h_0(n) = \begin{cases} \frac{1}{\sqrt{M}} & \text{for } n \in \{0, 1, \ldots, M-1\} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Clearly this scaling vector is 1-regular. From Eqn. 2 and Eqn. 1 it can be shown that every scaling vector is necessarily 1-regular. For all multiplicities M, K-regular scaling vectors can be constructed [3], and the corresponding WTFs are called K-regular WTFs.

2.2 Cosine-Modulated FIR Filter Banks

Cosine-modulated FIR filter banks are a special class of unitary FIR filter banks, where the analysis filters $h_i(n)$, are all cosine-modulates of a lowpass linear-phase prototype filter $g(n)$ [17]. The fundamental idea behind cosine-modulated filter banks is the following: in an M-channel filter bank, the analysis and synthesis filters are meant to approximate ideal M-th band filters, shown in Fig. 2. The passbands of these filters occupy adjacent frequency channels π/M apart. Given a real, prototype filter $g(n)$, with passband in $[-\frac{\pi}{2M}, \frac{\pi}{2M}]$, if we modulate it by $\cos\left((2i+1)\frac{\pi}{2M}n + \epsilon_i\right)$, where ϵ_i is an arbitrary phase, the passband of the resultant filter occupies the desired bands in Fig. 2. Can an FIR prototype filter be chosen so that the resultant filters form a perfect reconstruction filter bank? This question has been answered in the affirmative independently by Malvar [18] and Koilpillai [17, 24, 19, 20]. We will use the complete parameterization of cosine-modulated filter banks of length $N = 2Mm$ in [17] to obtain our explicit parameterization of cosine-modulated WTFs.

In this paper we will use the modulation in [17]. That is,

$$h_i(n) = c_{i,n} g(n) \tag{6}$$

where $g(n)$ is an even-symmetric prototype filter of length $N = 2Mm$ for some non-negative integer m and

$$c_{i,n} = \cos\left(\frac{\pi}{2M}(2i+1)(n - \frac{N-1}{2}) + \tau_i\right). \tag{7}$$

The phase factor τ_i can be taken to be $(-1)^i \frac{\pi}{4}$. In order to see how unitariness of the cosine-modulated filter bank is reflected on the prototype filter it is convenient to define the polyphase components of the prototype filter.

$$\begin{aligned} G(z) &= \sum_{n=0}^{2M-1}\left[\sum_{l=0}^{m-1} g(2Ml+n)z^{-2Ml}\right]z^{-n} \\ &= \sum_{n=0}^{2M-1} z^{-n} G_n(z^{2M}). \end{aligned} \tag{8}$$

Each of $G_n(z)$ is an FIR sequence of length m, and there are $2M$ of them accounting for the length $2Mm$ of $g(n)$. A cosine-modulated filter bank is unitary iff for $n \in \{0, \ldots, M-1\}$ ([17])

$$G_n(z^{-1})G_n(z) + G_{M+n}(z^{-1})G_{M+n}(z) = \frac{2}{M}. \tag{9}$$

From the even symmetry of $G(z)$ it follows that

$$G_n(z) = z^{-(m-1)}G_{2M-1-n}(z^{-1}). \tag{10}$$

Therefore, if we define $J = \lfloor \frac{M}{2} \rfloor$, for even M Eqn. 9 need hold only for $n \in \{0, \ldots, J-1\}$. For odd M, however, one additionally requires

$$G_J(z)G_J(z^{-1}) = \frac{1}{M}. \tag{11}$$

The most general FIR solution to the Eqn. 11 is $G_J(z) = \pm\frac{1}{\sqrt{M}}z^{-k}$ for some integer k. As for Eqn. 9, each of the J pair of filters $\{G_n(z), G_{M+n}(z)\}$ are characterized by m angle parameters $\theta_{n,l}$, $l \in \{0, \ldots, m-1\}$ [17]. Indeed, one has the following lattice factorization of these pairs:

$$\begin{bmatrix} G_n(z) \\ G_{n+M}(z) \end{bmatrix} = \sqrt{\frac{2}{M}} \prod_{l=m-1}^{1} \left\{ \begin{bmatrix} \cos(\theta_{n,l}) & \sin(\theta_{n,l}) \\ \sin(\theta_{n,l}) & -\cos(\theta_{n,l}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \right\} \begin{bmatrix} \cos(\theta_{n,0}) \\ \sin(\theta_{n,0}) \end{bmatrix} \tag{12}$$

Therefore, a unitary filter bank is parameterized by $Jm = \lfloor \frac{M}{2} \rfloor \approx N/4$ parameters $\theta_{n,l}$.

3 Cosine-Modulated Wavelet Tight Frames

The Wavelet Tight Frames Theorem states that an FIR unitary filter bank gives rise to a WTF iff the linear constraint in Eqn. 2 (or equivalently $H_0(1) = 1$) holds. Since the cosine-modulated filter bank is expressed in terms of the polyphase components of the prototype fitler, and the linear constraint is directly on the lowpass filter, it is convenient to express the filters in terms of the polyphase components. The modulation vector satisfies, $c_{i,n+2lM} = (-1)^l c_{i,n}$ and therefore

$$\begin{aligned} H_i(z) &= \sum_{n=0}^{2M-1} c_{i,n} z^{-n} \left[\sum_{l=0}^{m-1} (-1)^l g(2Ml+n) z^{-2Ml} \right] \\ &= \sum_{n=0}^{2M-1} c_{i,n} z^{-n} G_n(-z^{2M}). \end{aligned} \tag{13}$$

For $H_0(1)$ to be equal to one, we therefore require (from Eqn. 13)

$$\sum_{n=0}^{2M-1} c_{0,n} G_n(-1) = \sqrt{M}. \tag{14}$$

The polyphase components $G_n(z)$ has to merely satisfy the *single* constraint above. An interesting result that occurs is that this single constraint decouples into J constraints one for each polyphase component pair. Moreover, despite the nonlinear dependence of a polyphase pair on the lattice angles generating it, the constraint reduces to a simple linear constraint on the angles. Also it turns out that while one has to consider the case of odd and even M separately for cosine-modulated filter banks, the WTF conditions on each lattice is one clean formula independent of the parity of M.

We first evaluate $G_n(-1)$ in terms of the lattice angle parameters. For $n \in \{0, 1, \ldots, M-1\}$, it is easy to see that $G_n(-1)$, $G_{M+n}(-1)$, $G_{M-1-n}(-1)$ and $G_{2M-1-n}(-1)$ depend on the same set of angle parameters. For all M Eqn. 12 implies

$$\begin{aligned} \begin{bmatrix} G_n(-1) \\ G_{n+M}(-1) \end{bmatrix} &= \sqrt{\frac{2}{M}} \prod_{l=m-1}^{1} \left\{ \begin{bmatrix} \cos(\theta_{n,l}) & -\sin(\theta_{n,l}) \\ \sin(\theta_{n,l}) & \cos(\theta_{n,l}) \end{bmatrix} \right\} \begin{bmatrix} \cos(\theta_{n,0}) \\ \sin(\theta_{n,0}) \end{bmatrix} \\ &= \sqrt{\frac{2}{M}} \begin{bmatrix} \cos\left(\sum_{l=0}^{m-1} \theta_{n,l} \right) \\ \sin\left(\sum_{l=0}^{m-1} \theta_{n,l} \right) \end{bmatrix} \stackrel{\text{def}}{=} \sqrt{\frac{2}{M}} \begin{bmatrix} \cos(\Theta_n) \\ \sin(\Theta_n) \end{bmatrix} \end{aligned} \tag{15}$$

Also from Eqn. 10

$$\begin{aligned} \begin{bmatrix} G_{2M-1-n}(-1) \\ G_{M-1-n}(-1) \end{bmatrix} &= \begin{bmatrix} (-1)^{(m-1)} G_n(-1) \\ (-1)^{(m-1)} G_{M+n}(-1) \end{bmatrix} \\ &= \sqrt{\frac{2}{M}} \begin{bmatrix} (-1)^{(m-1)} \cos(\Theta_n) \\ (-1)^{(m-1)} \sin(\Theta_n) \end{bmatrix}. \end{aligned} \tag{16}$$

Together we have

$$\begin{bmatrix} G_n(-1) \\ G_{M+n}(-1) \\ G_{M-1-n}(-1) \\ G_{2M-1-n}(-1) \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \cos(\Theta_n) \\ \sin(\Theta_n) \\ (-1)^{(m-1)}\sin(\Theta_n) \\ (-1)^{(m-1)}\cos(\Theta_n) \end{bmatrix} \tag{17}$$

For odd M, $G_J(z) = \pm\sqrt{\frac{1}{M}} z^{-k}$, and from the even symmetry of $g(n)$, $G_{M+J}(z) = \pm\sqrt{\frac{1}{M}} z^{-(m-1-k)}$. Therefore

$$\begin{bmatrix} G_J(-1) \\ G_{M+J}(-1) \end{bmatrix} = \pm\sqrt{\frac{1}{M}} \begin{bmatrix} (-1)^k \\ (-1)^{(m-1-k)}. \end{bmatrix} \tag{18}$$

Finally, if we define define $\alpha_n = \frac{\pi}{2M}(n - Mm + \frac{1}{2}) + \frac{\pi}{4}$ it is easy to verify that

$$\begin{bmatrix} c_{0,n} \\ c_{0,n+M} \\ c_{0,M-1-n} \\ c_{0,2M-1-n} \end{bmatrix} = \begin{bmatrix} \cos(\alpha_n). \\ -\sin(\alpha_n), \\ (-1)^{(m-1)}\cos(\alpha_n), \\ (-1)^{(m-1)}\sin(\alpha_n). \end{bmatrix} \tag{19}$$

Also for odd M

$$\begin{bmatrix} c_{0,J} \\ c_{0,J+M} \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{2}(1-m)) \\ -\sin(\frac{\pi}{2}(1-m)). \end{bmatrix} \tag{20}$$

We consider the case of even and odd M separately. When M is even the one has a cosine-modulated WTF iff

$$\begin{aligned}
\sqrt{M} &= \sum_{n=0}^{2M-1} c_{0,n} G_n(-1) \\
&= \sqrt{\frac{2}{M}} \left\{ \sum_{n=0}^{J-1} \cos(\alpha_n)\cos(\Theta_n) - \sin(\alpha_n)\sin(\Theta_n) + \cos(\alpha_n)\sin(\Theta_n) + \sin(\alpha_n)\cos(\Theta_n) \right\} \\
&= \sqrt{\frac{2}{M}} \left\{ \sum_{n=0}^{J-1} \cos(\alpha_n + \Theta_n) + \sin(\alpha_n + \Theta_n) \right\} \\
&= \frac{2}{\sqrt{M}} \sum_{n=0}^{J-1} \sin(\frac{\pi}{4} + \alpha_n + \Theta_n),
\end{aligned} \tag{21}$$

or equivalently

$$\sum_{n=0}^{J-1} \sin(\frac{\pi}{4} + \alpha_n + \Theta_n) = \frac{M}{2} = J. \tag{22}$$

Clearly each of the J sinusoids on the left hand side of the equation has to be one and therefore

$$\Theta_n = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1) \tag{23}$$

For odd M there are two additional terms in the expression $\sum_{n=0}^{2M-1} G_n(-1)c_{0,n}$, corresponding to $n = J$ and $n = J + M$. In this case one has a WTF iff

$$\sqrt{M} = \sqrt{\frac{2}{M}} \left\{ \sum_{n=0}^{J-1} \cos(\alpha_n + \Theta_n) + \sin(\alpha_n + \Theta_n) \right\} + c_{0,J} G_J(-1) + c_{0,M+J} G_{M+J}(-1) \tag{24}$$

For fixed m, since $c_{0,J}$ or $c_{0,M+J}$ is zero, and since $G_J(-1)$ and $G_{M+J}(-1)$ are of the form $\pm\sqrt{\frac{1}{M}}$

$$\sum_{n=0}^{J-1} \sin(\frac{\pi}{4} + \alpha_n + \Theta_n) = \frac{M \mp 1}{2}. \tag{25}$$

Clearly there is no solution if the right hand side is $J+1$. This fixes the sign of $G_J(z)$ and $G_{J+M}(z)$. Moreover, in this case since each of the sinusoids on the left hand side is one Eqn. 23 holds once again. We have therefore proved the following theorem.

Theorem 1 *Cosine-modulated multiplicity M WTFs exist for all M. All such WTFs with scaling function and wavelets supported in $[0, \frac{2Mm-1}{M-1})$ are parameterized by $J(m-1)$ free parameters or Jm angles $\theta_{n,l}$, $n \in \{0,\ldots,J-1\}$ and $l \in \{0,\ldots,m-1\}$ such that $\sum_l \theta_{n,l} = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1)$.*

Remark: The WTF was obtained by imposing one condition, viz., Eqn. 2 on the cosine-modulated filter bank that has Jm parameters. Therefore one expects the WTF to have $Jm-1$ parameters, not $J(m-1)$ as Theorem 1 shows. The linear constraint (on h_0) interestingly becomes a linear constraint on the angles of each of the J lattices (recall that the angles are related to h_0 non-linearly). For odd M one additional has the sign of the delays $G_J(z)$ and $G_{J+M}(z)$ constrained such that $G_J(-1)c_{0,J} + G_{M+J}(-1)c_{0,M+J} = \sqrt{\frac{1}{M}}$). Why does this happen? From the the quadratic condition (Eqn. 1), by taking the $\mathcal{Z}$-transform on both sides we get

$$\begin{aligned} 1 &= \mathcal{Z}(\sum_k h_0(k)h_0(k+Ml)) \\ &= [\downarrow M]\, H_0(z)H_0(z^{-1}) \end{aligned} \tag{26}$$

and therefore

$$\frac{1}{M}\sum_{k=0}^{M-1} \left|H_0(e^{\imath(\omega+\frac{2\pi k}{M})})\right|^2 = 1. \tag{27}$$

Since $H_0(e^{\imath 0}) = \sqrt{M}$, we have for $k \in \{1,\ldots,M-1\}$

$$H_0(\frac{2\pi k}{M}) = 0. \tag{28}$$

From the symmetry of the roots of unity, the $(M-1)$ complex constraints reduce to $M-1$ real constraints. Together with Eqn. 2 there are M linear-constraints. And with linear-phase this reduces to $M/2 = J$ constraints.

Two examples of cosine-modulated WTFs, one each for odd and even M respectively follow.

Example 1 $M = 2$ **and** $N = 4$. Since $N = 2M$, there are no free parameters. Since $J = 1$, there is precisely one lattice with one angle parameter which is given (from Eqn. 23) by

$$\Theta_0 = \theta_{0,0} = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1) = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\,\pi}{8} \tag{29}$$

The prototype filter is given by

$$g = \sqrt{\frac{2}{M}} \left[\begin{array}{cccc} \cos(\theta_{0,0}) & \sin(\theta_{0,0}) & \sin(\theta_{0,0}) & \cos(\theta_{0,0}) \end{array}\right] \tag{30}$$

The corresponding scaling and wavelet vectors (from Eqn. 6) are given by

$$h_0 = \left[\begin{array}{cccc} \frac{1}{2\sqrt{2}} & \frac{\sqrt{2}+1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{\sqrt{2}-1}{2\sqrt{2}} \end{array}\right] \quad \text{and} \quad h_1(n) = \left[\begin{array}{cccc} -\frac{\sqrt{2}-1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{\sqrt{2}+1}{2\sqrt{2}} \end{array}\right] \tag{31}$$

The scaling function, wavelet, their Fourier transforms and the discrete-time Fourier transform of

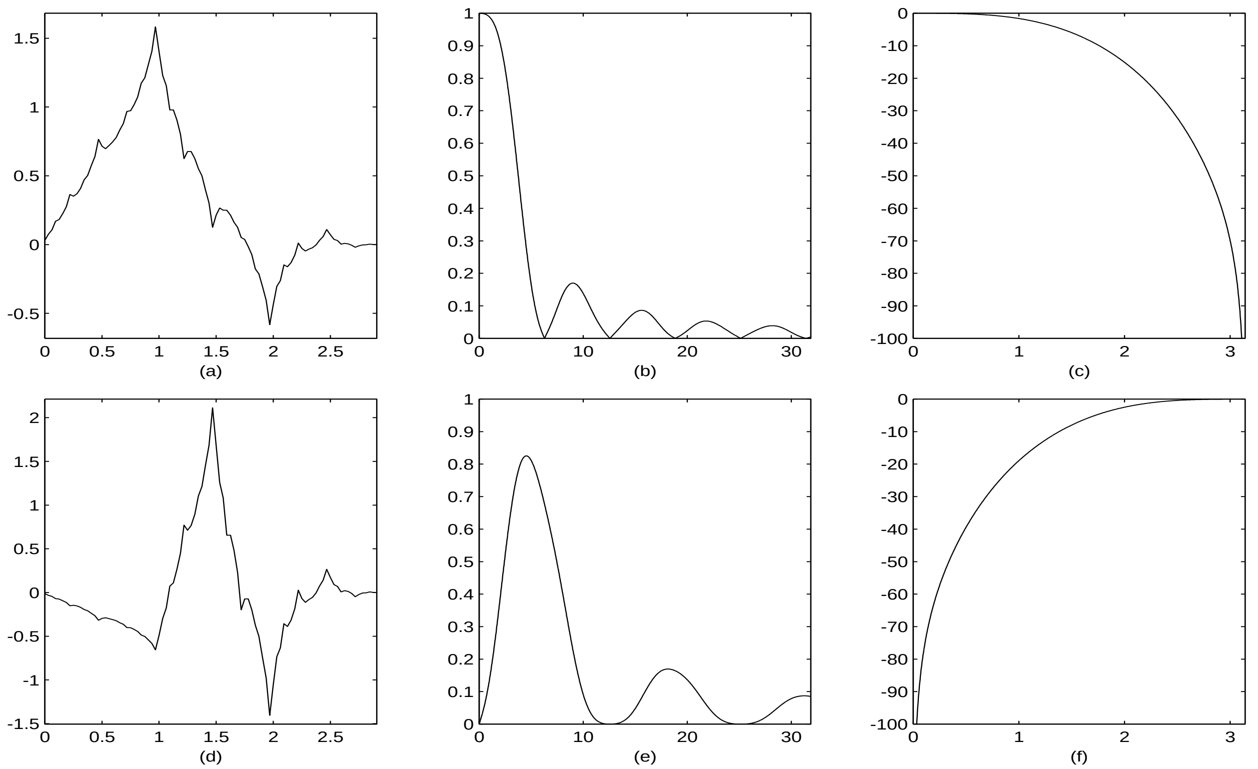

Figure 3: Example 1: $M = 2$, $N = 4$. (a) Scaling Function $\psi_0(t)$ (b) Fourier transform $\hat{\psi}_0(\omega)$ (c) $20\log|H_0(\omega)|$ (d) Wavelet $\psi_1(t)$ (e) Fourier Transform $\hat{\psi}_0(\omega)$ and (f) $20\log|H_1(\omega)|$

the corresponding filters are shown in Fig. 3. The scaling function and wavelet (both of which are supported in $[0,3)$) exhibit "kinks" in their graph, while their Fourier transforms are well behaved. These functions are not as smooth as the corresponding length 4 example in [4] (where the WTF is not cosine-modulated).

Example 2 $M = 5$ **and** $N = 10$ Again since $N = 2M$, there are no free parameters. In this case since $J = \lfloor \frac{M}{2} \rfloor = 2$, there are two lattices each with one angle parameter given by (from Eqn. 23)

$$\Theta_0 = \theta_{0,0} = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1) = \frac{\pi}{2} - \frac{\pi}{20} = \frac{9\,\pi}{20}$$

$$\Theta_1 = \theta_{0,0} = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1) = \frac{\pi}{2} - \frac{3\,\pi}{20} = \frac{7\,\pi}{20}$$

Since M is odd, $G_J(z)$ is an arbitrary delay. We will choose $g(J) = g(J+M) = \frac{1}{\sqrt{2}}$. While for a cosine-modulated filter bank it is irrelevant whether these two numbers are $+\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$, it is *necessary* that they be both positive for a WTF. The prototype filter g is given by

$$\sqrt{\frac{2}{M}}\left[\begin{array}{cccccccccc} \cos(\theta_{0,0}) & \cos(\theta_{1,0}) & \frac{1}{\sqrt{2}} & \sin(\theta_{1,0}) & \sin(\theta_{0,0}) & \sin(\theta_{0,0}) & \sin(\theta_{1,0}) & \frac{1}{\sqrt{2}} & \cos(\theta_{1,0}) & \cos(\theta_{0,0}) \end{array}\right].$$

The scaling and wavelet vectors are readily obtained (in Section 3.1 we give formulas for them). The scaling function, wavelets, their Fourier transforms and the Fourier transforms of the corresponding filters are shown in Fig. 4.

3.1 Explicit Formula for WTFs with $N = 2M$

If $N = 2M$ (as in the two cases above) the cosine-modulated WTF is unique since there are no free parameters. Therefore an explicit formula for the scaling and wavelet vectors can be obtained. This has important consequences in transform coding applications (especially image coding), where very long filters would exhibit ringing artifacts and one has to keep N as small as possible, while simultaneously having M large [20].

When $N = 2M$ (and M even) it is easy to see that the prototype filter is of the form

$$\left[\begin{array}{c} g(0) \\ \cdots \\ g(J-1) \\ g(J) \\ \cdots \\ g(M-1) \\ g(M) \\ \cdots \\ g(M+J) \\ \cdots \\ g(2M-1) \end{array}\right] = \sqrt{\frac{2}{M}} \left[\begin{array}{c} \cos(\Theta_0) \\ \cdots \\ \cos(\Theta_{J-1}) \\ \sin(\Theta_{J-1}) \\ \cdots \\ \sin(\Theta_0) \\ \sin(\Theta_0) \\ \cdots \\ \sin(\Theta_{J-1}) \\ \cdots \\ \cos(\Theta_0) \end{array}\right]. \tag{32}$$

Therefore, for $n \in \{0, \ldots, J-1\}$, $g(n) = \cos\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Also for $n \in \{0, 1, \ldots, J-1\}$, $g(J+n) = \sin(\frac{\pi}{2} - (2(J-1-n)+1)\frac{\pi}{4M}) = \cos\left(\frac{\pi}{2} - (2(J+n)+1)\right)\frac{\pi}{4M})$. This implies that for $n \in \{0,\dots,M-1\}$, $g(n) = \cos\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Similarly it can be shown that for $n \in \{0,\dots,M-1\}$, $g(M+n) = \sin\left(\frac{\pi}{2} - (2n+1)\frac{\pi}{4M}\right)$. Equivalently, $g(M+n) = \cos\left(\frac{\pi}{2} - (2(M+n)+1)\frac{\pi}{4M}\right)$. Putting it all together we have the following simple formula for the prototype filter:

$$g(n) = \sin\left(\frac{\pi}{4M}(2n+1)\right) \tag{33}$$

One can similarly do an analysis for the case when M is odd and show that the prototype filter is given by Eqn. 33. Therefore the scaling and wavelet vectors are given by

$$\begin{aligned} h_i(n) &= g(n)c_{i,n} = g(n)\cos\left(\frac{\pi}{2M}(2i+1)(n-\frac{2M-1}{2}) + (-1)^i\frac{\pi}{4}\right) \\ &= g(n)(-1)^i \sin\left(\frac{\pi}{2M}(2i+1)(n+\frac{1}{2}) + \frac{\pi}{4}(-1)^i\right) \\ &= \sqrt{\frac{1}{2M}}\left[\sin\left(\frac{\pi(i+1)(n+.5)}{M} - (-1)^i\frac{\pi}{4}\right) - \sin\left(\frac{\pi i(n+.5)}{M} - (-1)^i\frac{\pi}{4}\right)\right]. \end{aligned} \tag{34}$$

Theorem 2 *Scaling/wavelet vectors in a cosine-modulated WTFs with $N = 2M$ are given by Eqn. 34.*

3.2 Orthonormality of Cosine-Modulated Wavelet Tight Frames

All examples of cosine-modulated WTFs that we construct in this paper are verified to be orthonormal based on the characterization of orthonormality of multiplicity M WTFs in [3], and extension of the ideas in [27]. Based on our experiments we make the following conjecture.

Conjecture 1 *Every multiplicity M, cosine-modulated wavelet tight frame is necessarily an orthonormal wavelet basis.*

4 Higher Regularity Cosine-Modulated WTFs

In many applications using wavelets, especially in numerical analysis and the solution of partial differential equations, it is desirable to have as many moments of the wavelets as possible vanish. Usually this is not required in signal processing. However, when signals have a local polynomial behavior this property of the wavelets is useful. There is yet another reason that this property *may* be useful in signal processing: if filter banks are iterated in a tree structure, regularity is known to give reasonable iterated filter responses [4]. This section discusses the design of such regular cosine-modulated WTFs and some of the difficulties associated with it.

The basic idea is to impose the regularity condition on the parameterization of cosine-modulated WTFs. By solving the set of non-linear equations associated with these constraints one obtains regular WTFs. For a given K, there is a smallest $N = 2Mm$ such that the scaling vector can be K-regular. Such scaling vectors will be called *maximally regular* cosine-modulated scaling vectors. The length, N_{min}, of maximally regular scaling vectors grows rapidly with K. For K-regularity (from Eqn. 4), we want for $k \in \{1, \ldots, K-1\}$,

$$\left(\frac{d}{dz^{-1}}\right)^k H_0(z) = 0 \tag{35}$$

for $z = e^{\imath 2\pi l/M}$, $l \in \{1, \ldots, M-1\}$. From the symmetry of the roots of unity, these complex constraints for each k reduce to a set of $M-1$ real constraints. Hence there are $(K-1)(M-1)$ constraints. Since $J(m-1)$, the number of parameters has to be atleast as many as the number of constraints, for the maximally regular case

$$N_{min} = 2M\left(\left\lceil \frac{(K-1)(M-1)}{J} \right\rceil + 1\right) \tag{36}$$

This is in contrast to the case of general (not cosine-modulated case) WTF case, where maximally regular scaling vectors are of length $N = MK$ [3].

For design purposes it is convenient to write Eqn. 35 in a form that makes the $(K-1)(M-1)$ constraints on h_0 transparent: for $l \in \{1, \ldots, M-1\}$ and $k \in \{1, \ldots, K-1\}$ we have [3]

$$\eta_{k,0} \stackrel{\text{def}}{=} \sum_n (Mn)^k h_0(Mn) = \sum_n (Mn+l)^k h_0(Mn+l) \stackrel{\text{def}}{=} \eta_{k,l} \tag{37}$$

We now discuss a simple example of regular cosine-modulated WTF design.

Example 3 Let $M = 2$ and $K = 2$, in which case $N_{min} = 8$. There is one free parameter and one the following non-linear equation (which can be solved analytically).

$$\sum_n h_0(2n+1) = \sum_n h_0(n). \tag{38}$$

Firstly from Eqn. 23

$$\Theta_0 = \theta_{0,0} + \theta_{0,1} = \frac{7\pi}{8}. \tag{39}$$

It can be easily shown (after straightforward algebraic manipulations) that Eqn. 38 in conjunction with Eqn. 39 is equivalent to

$$\sin(2\theta_{0,1}) - \cos(2\theta_{0,1})) = (1 + \sin(\frac{\pi}{4}))/2.$$

Therefore, there are two solutions to $\theta_{0,1}$ given by

$$\theta_{0,1} = \frac{1}{2}\left[\frac{\pi}{4} + \arcsin(\frac{\sqrt{2}+1}{4})\right] \text{ or } \frac{1}{2}\left[\frac{\pi}{4} + \pi - \arcsin(\frac{\sqrt{2}+1}{4})\right].$$

Table 2: Scaling and Wavelet Vectors : $M = 2$, $K = 2$, $N = 8$

	Case 1		Case 2	
n	$h_0(n)$	$h_1(n)$	$h_0(n)$	$h_1(n)$
0	0.12846511606780	0.31014222550271	0.01169992687716	0.02824612214561
1	-0.25836553020608	-0.62374956707322	0.02353057315103	0.05680782883163
2	-0.27019617647995	0.11191892079935	0.41036121942490	-0.16997718255776
3	0.54341116509056	-0.22508827452548	0.82530726844767	-0.34185346371612
4	0.22508827452548	0.54341116509056	0.34185346371612	0.82530726844767
5	0.11191892079935	0.27019617647995	-0.16997718255776	-0.41036121942490
6	0.62374956707322	-0.25836553020608	-0.05680782883163	0.02353057315103
7	0.31014222550271	-0.12846511606780	0.02824612214561	-0.01169992687716

where arcsin is assumed to return a value in $[-\pi/2, \pi/2]$. Scaling and wavelet vectors corresponding to the two cases are given in Table 2. Fig. 5 shows that scaling function and wavelet in Case 2 are smoother than the corresponding functions in Case 1. The Fourier transforms also show that the scaling function and wavelet in Case 1 do not approximate ideal lowpass and bandpass filters respectively as well as Case 2. A similar comment applies to the discrete-time Fourier transform of the scaling and wavelet filters. The example above illustrates a number of features of K-regular cosine-modulated WTFs. Firstly, there may be multiple solutions of maximally regular cosine-modulated scaling vectors (that are not distinct spectral factors - i.e., do not have the same autocorrelation function). This is in contrast to general WTFs where all maximally regular scaling vectors (see [4] for the multiplicity 2 case and [3] for the multiplicity M case) are necessarily spectral factors and share the same autocorrelation function. Secondly, that regularity of the scaling vector does not imply "smoothness" the scaling function. The example emphasizes that numerical designs of K-regular cosine-modulated wavelet bases may not be smooth. However, it also suggests that (see (c) and (i) in Fig. 5) if one initializes the optimization routine with a good lowpass filter, then there is a reasonable chance that the numerical design would be smooth. A good initial guess for the angles, it turns out, is to set $\theta_{n,l} = \pi/2$, for $l \in \{1, \ldots, m-1\}$ and all n. Another fact that is not so widely appreciated also follows: an increase in the regularity of the scaling vector does not imply a corresponding increase in the smoothness of the scaling function. To illustrate this we have constructed a 3 regular multiplicity 2 cosine-modulated WTF, the scaling function and wavelets of which are given in Fig. 6 and the corresponding scaling and wavelet vectors in Table 3. Clearly, these functions are not as smooth as the 2 regular example in Example 1 (Case 2). If K-regularity

Table 3: Scaling and Wavelet vectors: $M = 2$, $K = 3$, $N = 12$

n	$h_0(n)$	$h_1(n)$	n	$h_0(n)$	$h_1(n)$
0	-0.03351073370344	0.01388060038504	6	0.16458796961057	0.39735050843730
1	0.05624428949080	-0.02329714751313	7	0.11772979770095	0.28422487430507
2	0.16566827346957	0.39995859266518	8	0.67128899860178	-0.27805700749271
3	-0.27805700749271	-0.67128899860178	9	0.39995859266518	-0.16566827346957
4	-0.28422487430507	0.11772979770095	10	0.02329714751313	0.05624428949080
5	0.39735050843730	-0.16458796961057	11	0.01388060038504	0.03351073370344

of the scaling vector (and not the smoothness of the scaling function) is critical then the method gives a solution. We do not have a proof but make the following conjecture:

Conjecture 2 *There exist multiplicity M, K-regular wavelet tight frames for all M and K.*

Since in most numerical analysis applications only low order of K is required, for $M \in \{2, \ldots, 8\}$, 2 regular prototype filters are given in Table 4. Table 4 was generated by numerically solving the non-linear equations in Eqn. 35. By starting with different initializations, the problem was solved repeatedly and until "smooth" solutions were found. The prototype filters given correspond to these "smooth" solutions. Moreover since the prototype filters are even symmetric, $g(n)$ is only given for $n \in \{0, 1, \ldots, N/2 - 1\}$. The frequency responses of the filters are shown in Fig. 7. For $M = 2$, $N_{min} = 2MK$ and therefore are $K - 1$ non-linear equations for exactly $K - 1$ unknowns. Also for all odd M $N_{min} = 2M(2K - 1)$ and once again there are $(M - 1)(K - 1)$ equations with precisely $J(2K - 2) = (M - 1)(K - 1)$ parameters. For these values of M it is possible to obtain smooth K-regular solutions. However for even M $(M > 2)$ it is difficult to find smooth K-regular solutions because there are more unknowns than there are equations.

A logical approach to take care of this situation is to use the extra degrees of freedom to impose *some* of the equations corresponding to a regularity of $K + 1$. This approach was found to give very poor solutions. It seems to be the case that imposing some of the $(M - 1)$ equations corresponding to increasing the regularity from K to $K + 1$ does more harm than not imposing it at all. We now introduce a technique, the moments minimization method to solve this problem.

4.1 Moments Minimization Method

In coding applications it is desirable to have short filters. If the length is fixed what is the best that can be achieved in terms of regularity? The moments minimization method provides an answer to

Table 4: Prototype filters $g(n)$ for K-regular Cosine-Modulated WTFs

$\boxed{K = 2}$

$M = 3,\ N = 18$	$M = 5,\ N = 30$	$M = 6,\ N = 36$	$M = 7,\ N = 42$	$M = 8,\ N = 48$
-0.00388862514471	-0.00301097038898	-0.00054944468846	-0.00246739083642	-0.00123930643393
0.00000000000000	-0.00201208359560	-0.01910742365573	-0.00237194122154	-0.00779774978931
-0.01478482519704	0.00000000000000	-0.00577566692183	-0.00107950489740	-0.01487487447880
-0.02767653677256	-0.00448172450143	0.02844407733543	0.00000000000000	-0.01932102524122
0.00000000000000	-0.01512586189495	0.01910967329910	-0.00191595817083	0.08973245101789
0.10522813154117	-0.02162484892812	-0.00265043105253	-0.00743504813986	0.06138828332885
0.31266437180164	-0.01722782422909	-0.01338305956067	-0.01402758013820	0.03296682908719
0.57735026918963	0.00000000000000	-0.00910367899718	-0.01803575190864	0.00750441807299
0.74621377262522	0.03837333703375	-0.00579716969829	-0.01735610151889	-0.01402667671272
	0.10863423950739	-0.02854997448027	-0.01164038179004	-0.02960269106783
$M = 4,\ N = 24$	0.20456111193383	-0.00910475083353	0.00000000000000	-0.03627575519340
0.00877038984401	0.32349005648916	0.06455768411718	0.02065991979828	-0.03352506774898
-0.01075359886307	0.44721359549996	0.13936757164327	0.05440415181523	-0.15570014852204
0.05457943758320	0.54181245657050	0.19273020820114	0.10253663567074	-0.14970925240107
0.01059371915934	0.58791824411149	0.31714293350835	0.15991677470121	-0.12515236873747
-0.01813390625490		0.48068967221677	0.22857379514531	-0.08493625413770
-0.01403215531070		0.54340809109609	0.30396352191435	-0.03716699040685
-0.07121961258650		0.55637746698764	0.37796447300923	0.01219222010045
0.02190387354986			0.43903678212524	0.07111424153313
0.16862395303534			0.47973571249855	0.14217546831856
0.31087426774399			0.49909818659237	0.44271260995028
0.62848508348218				0.46607316030963
0.68597973556552				0.48183430588546
				0.49107010469637

$\boxed{K = 3}$

$M = 2,\ N = 12$	$M = 3,\ N = 30$	
-0.01047051524186	-0.00002420872650	-0.01708533214152
-0.04042851803229	0.00000000000000	-0.02916098647507
-0.03544505570212	-0.00023932006300	0.00000000000000
0.13685965212863	0.00016871556489	0.10673613707477
0.50907256925186	0.00000000000000	0.32065980778007
0.84800595877688	-0.00166787045212	0.57735026918963
0.00090672612149	0.74249942047633	
	0.00000000000000	

this question. Instead of solving Eqn. 35, this method solves the following equivalent problem (see Eqn. 37):

$$\min_g \sum_{l=1}^{M-1} \sum_{k=1}^{M-1} |\eta_{k,0} - \eta_{k,l}|^2 . \tag{40}$$

The advantage of this formulation is that it can be solved for an arbitrary length scaling vector (unlike the earlier method of solving non-linear equations). This method of design in our experiments always gave smooth designs of WTFs. When this method is used to design WTFs with $N = 4M$ ($M > 3$), a surprising observation is made. All the solutions were smooth (notice that Eqn. 35 implies that they could only be 1-regular and hence at most differentiable once) and the scaling functions had *nearly identical* wave shape independent of M. Fig. 8 shows examples of such scaling functions when $M = 5$ and $M = 18$ respectively. Their remarkable similarity is apparent. Based on numerical experiments, one can obtain a simple approximate analytical formula for all scaling/wavelet vectors of a length $N = 4M$ that is obtained by the moment minimization method. The scaling functions and wavelets of WTFs generated by this formula are always smooth and virtually indistinguishable from those obtained by minimization.

4.1.1 Empirical Formula for WTFs with $N = 4M$

The prototype filter in this case is given by

$$g(n) = \sqrt{\frac{1}{2M}} \left[\cos(\beta_n) - \cos\left(\frac{\pi(2n+1)}{4M} \right) \right] \qquad \text{for } n \in \{0, \ldots, 4M-1\} \tag{41}$$

where the β_n is given by

$$\beta_n = \begin{cases} \gamma_M + \frac{\pi - 4\gamma_M}{2(M-1)} n & \text{for } n \in \{0, \ldots, M-1\} \\ \beta_{2M-1-n} & \text{for } n \in \{M, \ldots, 2M-1\} \\ \beta_{n-2M} & \text{for } n \in \{2M, \ldots, 4M-1\} \end{cases} \tag{42}$$

and γ_M is given by

$$0.4717 + e^{(-0.00032084024272M^3 + 0.01619976915653M^2 - 0.39479347799199M - 2.24633148545678)}.$$

The accuracy of this formula may be improved by making the dependence of β_n on n quadratic (or even higher order). However for most practical applications the above formulas with linear dependence should suffice.

The moments minimization method was introduced to take care of designing approximations to K-regular WTFs for all possible lengths N. Now consider the 2-regular example in Table 4 corresponding to $M = 4$. The scaling function corresponding to this design is shown in Fig. 9.

Consider the same design using the moments minimization method. From Eqn. 36 $N_{min} = 24$ and hence there are $Jm = 2 \times 3 = 6$ free parameters. Since only 3 parameters are required for 2-regularity there is one additional parameter. In the moments minimization method one minimizes $|\eta_{1,0} - \eta_{1,1}|^2 + |\eta_{1,0} - \eta_{1,2}|^2 + |\eta_{1,0} - \eta_{1,2}|^2 + |\eta_{2,0} - \eta_{2,1}|^2$. Notice that the last term could have been replaced by $|\eta_{2,0} - \eta_{2,2}|^2$ or $|\eta_{2,0} - \eta_{2,3}|^2$. The scaling function for this design is also shown in Fig. 9. The moments minimization method gives a smoother design. Table 5 shows the difference between partial moments in each case. While the moments minimization method gives a smoother design,

Table 5: Partial Moment Differences

	Regular Case	Moments Min. Case
$\eta_{1,0} - \eta_{1,1}$	-0.00000000330467	-0.07747566254995
$\eta_{1,0} - \eta_{1,2}$	-0.00000001680375	-0.02021563577220
$\eta_{1,0} - \eta_{1,3}$	0.00000000399227	-0.00557512157068
$\eta_{2,0} - \eta_{2,1}$	3.99093728958494	0.00349430819099

it has smeared the errors among the various moments. The reason is that all the moments are weighted uniformly. One can fix this problem by modifying the method to a weighted moments minimization method:

$$\min_{g} \sum_{l=1}^{M-1} \sum_{k=1}^{M-1} w_k \left| \eta_{k,0} - \eta_{k,l} \right|^2 . \tag{43}$$

where w_k is the weight sequence. If, for instance the j^{th} moment must be minimized definitely, then one may choose w_k such that $w_j << w_k, k \neq j$.

5 Optimal and Smooth Cosine-Modulated Wavelet Tight Frames

When wavelets are used in the representation of signals, it is more desirable to choose a wavelet that best approximates the signal, rather than a wavelet with a high degree of differentiability (and hence regularity of the scaling vector). In this case the theory of optimal (and robust) representation of signals in WTFs ([13, 12]) can be applied to the cosine-modulated case. The problem is to design the best multiresolution analysis (i.e., the scaling vector) (for an a priori fixed scale J) such that the error of projection of the signal onto scale J has minimum energy. The problem in general is hard to analyze, but under the realistic assumption of the signal being bandlimited, it simplifies considerably.

Let $\Omega = [-M^J\pi, M^J\pi]$. If a signal $f(t)$ is bandlimited to Ω, its approximation at scale J is given by

$$f(t) \approx Pf(t) = \sum_k \langle f, \psi_{0,J,k} \rangle \, \psi_{0,J,k}. \tag{44}$$

If the WTF is an orthonormal basis, and if $Qf = f - Pf$, denote the approximation error, (from orthonormality) minimizing the error, $\|Qf\|^2$, is the same as maximizing $\|Pf\|^2$, which is given by [12, 13],

$$\|Pf\|^2 = \frac{1}{2\pi} \int_\Omega \left|\hat{f}(\omega)\right|^2 \left|\hat{\psi}_0\left(\frac{\omega}{M^J}\right)\right|^2 d\omega \tag{45}$$

Since $\psi_0(t)$ is determined uniquely by h_0 and parameterized by $J(m-1)$ parameters, the optimal design problem becomes and an unconstrained optimization problem with that many parameters. An efficient numerical scheme for this optimization (that avoids explicit computation of the integrals in Eqn. 45) is given in [12].

We now give an application of the optimal design technique to design smooth WTFs. The essential intuitive idea behind our approach is that the optimal scaling function representing a smooth signal will tend to be smooth. Experimental evidence (tried for different values of M, N and different choices of $f(t)$) corroborates this intuition. Another interesting observation is that for smooth signals, the scaling function is robust to the specific choice of smooth signal (i.e., it does not matter what $f(t)$ is so long as it is smooth).

Example 4 In this example $M = 5$ and $N = 40$. The prototype filter corresponding to this design is given in Table 6. The scaling function and its Fourier transform is shown in Fig. 10. In this

Table 6: Prototype Filter of Optimal Design: $M = 5$ and $N = 40$

n	$g(n)$	n	$g(n)$	n	$g(n)$	n	$g(n)$
0	-0.00006668425535	5	0.00048662826448	10	-0.03366176844022	15	0.25344553542239
1	0.00011530060748	6	-0.00517687866088	11	-0.02009764274979	16	0.35471807435280
2	0.00000000000000	7	0.00000000000000	12	0.00000000000000	17	0.44721359549996
3	-0.00057346899740	8	-0.02574816803093	13	0.07219268096526	18	0.51756089391283
4	-0.00455654302741	9	-0.03325136666370	14	0.14946452131508	19	0.55782250408599

example it can be verified that the partial moments do not match up (even for the first moment) and hence the scaling function is *not* differentiable. However, for signal processing applications, the smoothness of the scaling function is adequate.

6 Towards a New Definition of Smoothness

Solving non-linear equations for K-regularity, the method of (weighted) moments minimization, and the optimal design, each has its own application niche. Moreover each method, under suitable conditions, gives smooth WTFs. All through this paper, the word *smooth* has been used loosely, appealing to "intuition" more so than reason. To motivate our definition, an understanding fundamentally why smoothness is needed is crucial. Our discussion, though made in the context of cosine-modulated filter banks and WTFs is equally applicable for the general filter bank and WTF design problems.

First consider the filter bank design problem. It is easy to show that the goal is ([17])

$$\min_g \int_{|\omega|>\frac{\pi}{2M}+\epsilon} |G(\omega)|^2 \, d\omega = \min_g \mathcal{E}(g), \tag{46}$$

where ϵ controls the transition width of the filters. If the filter bank is *not* used in a tree structure, then apparently there is no reason to consider smoothness, regularity etc. However, in most applications (like image/speech coding), filter banks *are* used in a tree structure. The impulse response of the effective filter along, say, the lowpass synthesis branch of such a tree is given by $H_0(z^{-1})H_0(z^{-M})\ldots H_0(z^{-M^{(J-1)}})$, where J is the depth of the tree. Since the reconstructed image is a superposition of such responses, it is desirable that this response is not too oscillatory. One way to control oscillations is to minimize the *total variation* of the impulse response.

We therefore define the depth-J smoothness, $\mathcal{TV}_J$ of a filter bank to be the total variation of its depth-J lowpass filter impulse response. That is

$$\mathcal{TV}_J = \sum_i |b_J(i) - b_J(i-1)| \tag{47}$$

where $b_J(i)$ is given by,

$$B_J(z) = H_0(z)H_0(z^M)\ldots H_0(z^{M^{(J-1)}}). \tag{48}$$

Therefore the filter bank design problem becomes one of minimizing some function of both $\mathcal{TV}_J$ and $\mathcal{E}$. The depth of the tree to be used in the application thus explicitly enters the design process. Many measures of smoothness (like the total variation of first differences of $B(z)$ etc.,) and combinations thereof can also be considered. An advantage of this method is that (locally) it tries to stretch out *kinks* in the impulse response and hence forces it to be smooth. For all filter bank designed using this method plots of $b(n)$ (for moderate to large J) were always smooth. Fig. 11 shows the impulse response of the lowpass filter h_0 designed by this method for a 2 channel, cosine-modulated filter bank with $N = 16$. The filter bank was assumed to be used in a 5 level tree. The design objective was (notice that other combinations of $\mathcal{TV}_5$ and $\mathcal{E}$ could also be used - in fact the choice should be dictated by the application).

$$\min_g \mathcal{TV}_5 \mathcal{E}. \tag{49}$$

A direct design based on minimizing $\mathcal{E}$ is also given for comparison. Both examples were designed without the WTF constraint, just to emphasize that this is a filter bank design approach (i.e., more general than a WTF design). Clearly the depth-5 smoothness of our new design is better. Moreover, the stopband attenuation is also better (which also means that the passband behavior is better because of unitariness of the filter bank). This improvement is achieved at the cost of an almost imperceptible increase in the transition width of the filter (see Fig. 11).

Now consider the design of smooth WTFs. This problem is clearly related (but we will presuppose a non-signal processing application). Here again, the total variation of the scaling function is defined to be the smoothness of the WTF. The smoothness of the WTF as defined above is exactly the same as the depth-∞ smoothness, $\mathcal{TV}_\infty$, of the associated unitary filter bank. Indeed,

$$\mathcal{TV}(\psi_0) = \int_{\mathrm{supp}(\psi_0)} \left| \frac{d\psi_0}{dt}(t) \right| dt. \tag{50}$$

7 Conclusion

This paper constructs and parameterizes compactly supported multiplicity M, cosine-modulated WTFs. In these WTFs the scaling vectors, which have lengths of the form $2Mm$, uniquely determine the wavelet vectors (unlike general multiplicity M WTFs). The paper obtains an explicit formula for scaling vectors of length $N = 2M$. In many applications one desires the WTF to be regular. Several numerical techniques for the construction of K-regular scaling vectors are introduced: solution of non-linear equations, the (weighted) moments minimization method, etc. Since K-regularity does not imply smoothness (degree of differentiability) of the WTF, a number of smooth K-regular designs (particularly for practically important case of small M and K) are given. An approximate formula for scaling/wavelet vectors of length $N = 4M$ obtained by moments minimization method (and giving smooth WTF) is also given. Optimal design of cosine-modulated WTFs for signal representation is described and adapted to the design of smooth WTFs. A novel definition of smoothness (that we think is relevant from a signal processing perspective) is proposed. This concept of smoothness is potentially very important in applications with tree structured filter banks. Preliminary designs of filter banks and WTFs based on this approach give good results This method minimizes undesirable oscillations in the scaling function. All cosine-modulated WTFs constructed are verified to be orthonormal bases. We conjecture that all cosine-modulated WTFs are orthonormal bases. This paper only describes cosine-modulated WTFs with $N = 2Mm$. A natural question can this form of restriction on N be released. To answer this question a comprehensive theory of modulated filter banks (with no assumptions on length of the filters, unitariness of the filter bank etc.) has been developed, and corresponding to these filters one obtains modulated filter banks and wavelet frames (tight and otherwise) [28].

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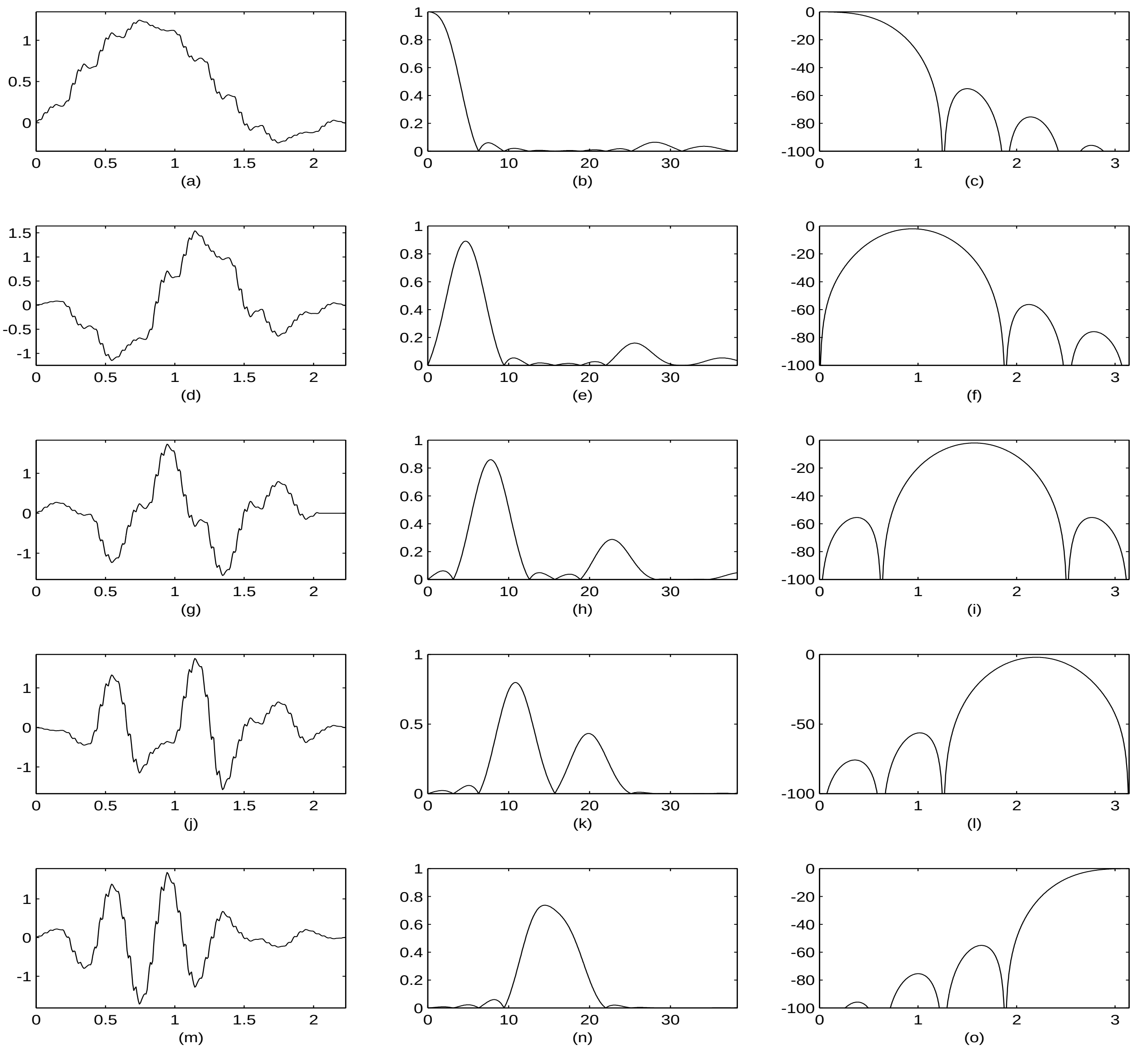

Figure 4: Example 2: $M = 5$, $N = 10$. (a) Scaling Function $\psi_0(t)$ (b) $\hat{\psi}_0(\omega)$ (c) $20 \log |H_0(\omega)|$ (d) Wavelet $\psi_1(t)$ (e) $\hat{\psi}_1(\omega)$ (f) $20 \log |H_1(\omega)|$ (g) Wavelet $\psi_2(t)$ (h) $\hat{\psi}_2(\omega)$ (i) $20 \log |H_2(\omega)|$ (j) Wavelet $\psi_3(t)$ (k) $\hat{\psi}_3(\omega)$ (l) $20 \log |H_3(\omega)|$ (m) Wavelet $\psi_4(t)$ (n) $\hat{\psi}_4(\omega)$ (o) $20 \log |H_4(\omega)|$

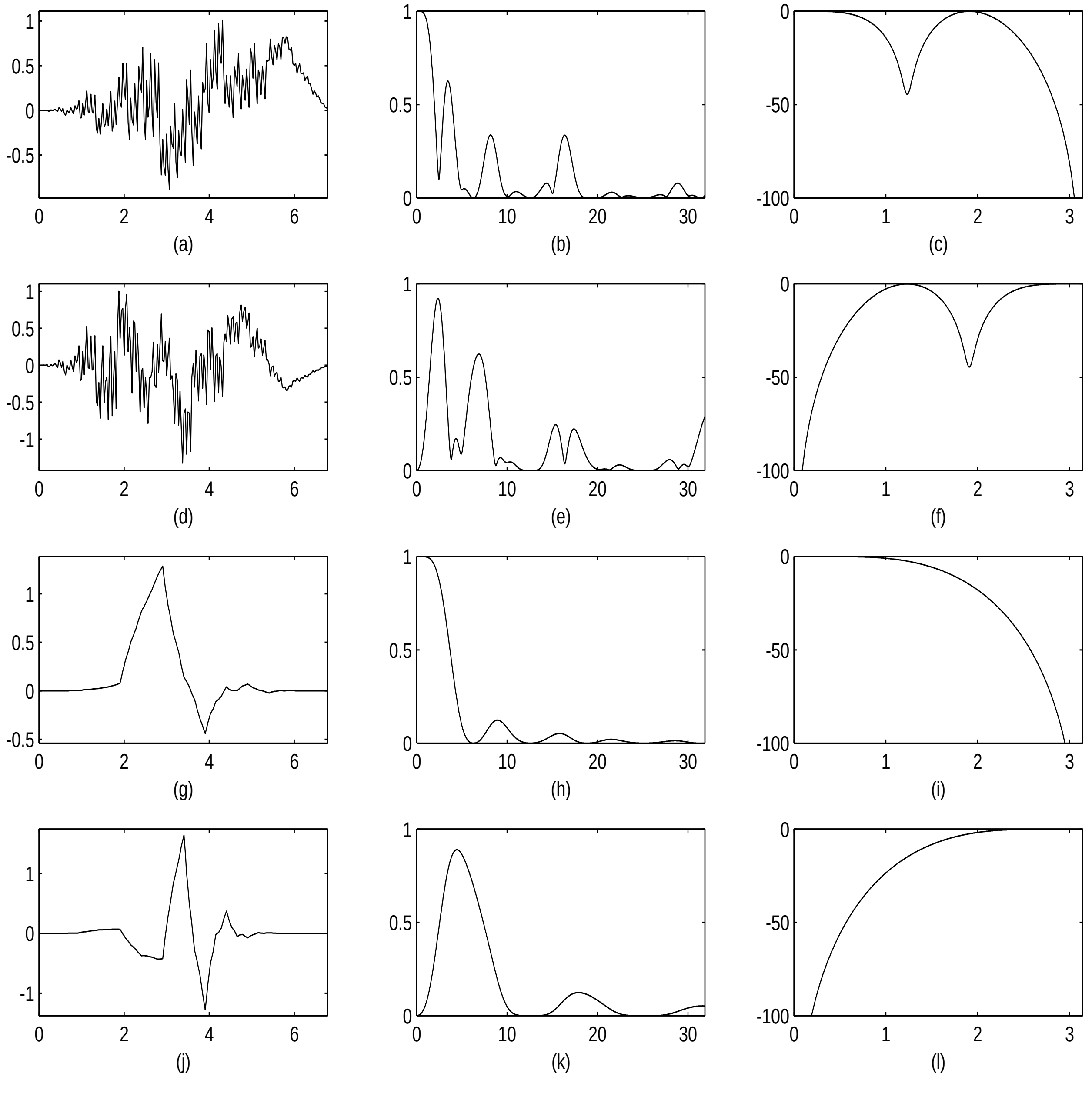

Figure 5: Example 3: $M = 2$, $K = 2$, $N = 8$. Case 1: (a) Scaling Function $\psi_0(t)$ (b) $\hat{\psi}_0(\omega)$ (c) $20\log|H_0(\omega)|$ (d) Wavelet $\psi_1(t)$ (e) $\hat{\psi}_1(\omega)$ (f) $20\log|H_1(\omega)|$ and Case 2: (g) Scaling Function $\psi_0(t)$ (h) $\hat{\psi}_0(\omega)$ (i) $20\log|H_0(\omega)|$ (j) Wavelet $\psi_1(t)$ (k) $\hat{\psi}_1(\omega)$ (l) $20\log|H_1(\omega)|$

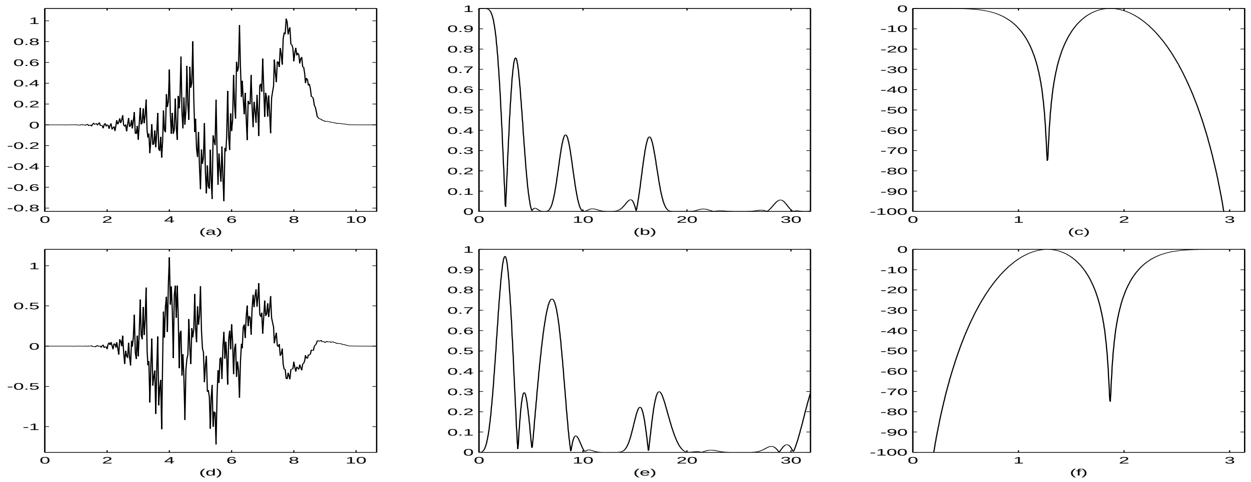

Figure 6: Example 3: $M = 2$, $K = 3$, $N = 12$. (a) Scaling Function $\psi_0(t)$ (b) $\hat{\psi}_0(\omega)$ (c) $20 \log |H_0(\omega)|$ (d) Wavelet $\psi_1(t)$ (e) $\hat{\psi}_1(\omega)$ (f) $20 \log |H_1(\omega)|$

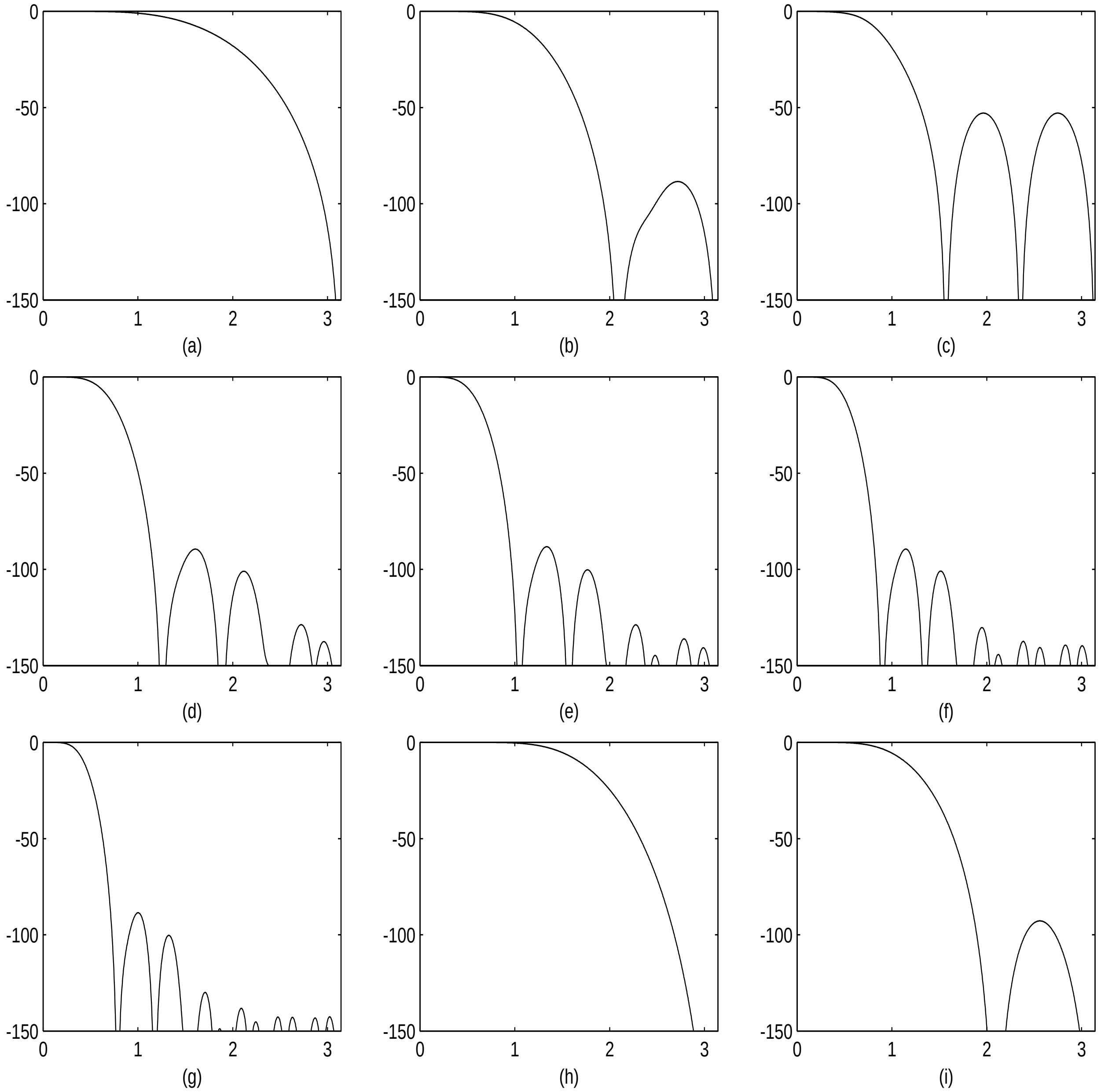

Figure 7: Fourier transform $H_0(\omega)$ of K-regular cosine-modulated scaling filters: I. $K = 2$ (a) $M = 2$ (b) $M = 3$ (c) $M = 4$ (d) $M = 5$ (e) $M = 6$ (f) $M = 7$ (g) $M = 8$ II. $K = 3$ (h) $M = 2$ (i) $M = 3$

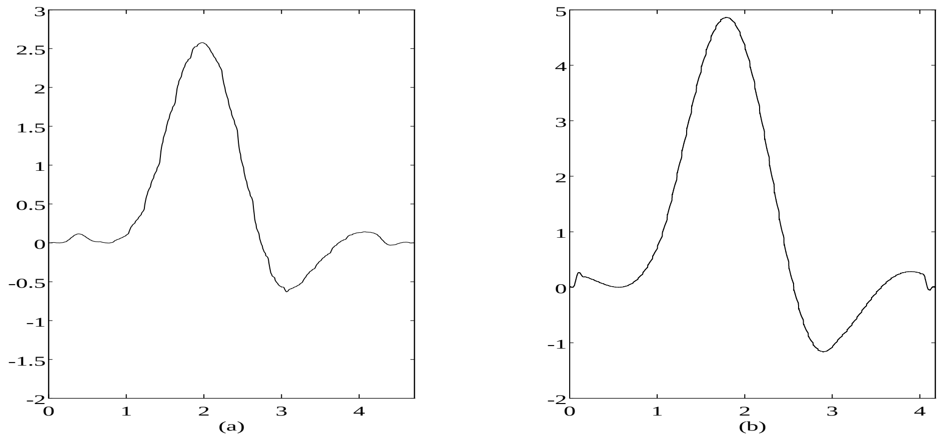

Figure 8: Moment Minimization Design: $N = 4M$ (a) Scaling Function $\psi_0(t)$ for $M = 5$ (b) Scaling Function $\psi_0(t)$ for $M = 18$

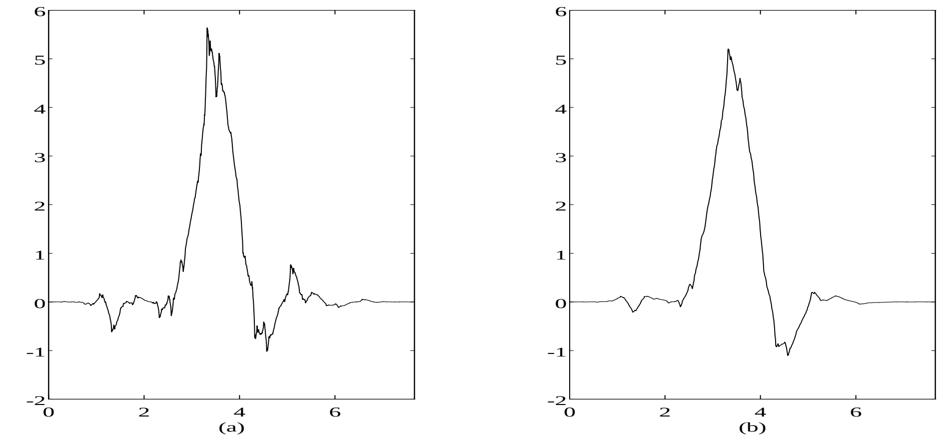

Figure 9: (a) Scaling Function $\psi_0(t)$ for $M = 4$, $K = 2$ (b) Scaling Function $\psi_0(t)$ for $M = 4$, moments minimization design

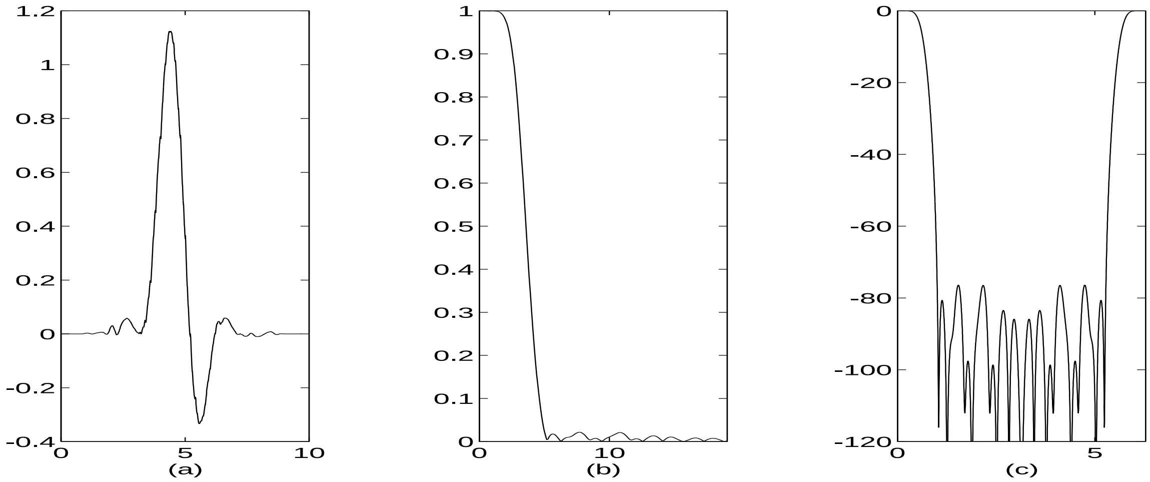

Figure 10: Optimal Design Example: (a) Scaling Function $\psi_0(t)$ (b) Fourier Transform $\hat{\psi}_0(\omega)$ and (c) Frequency response $H_0(\omega)$

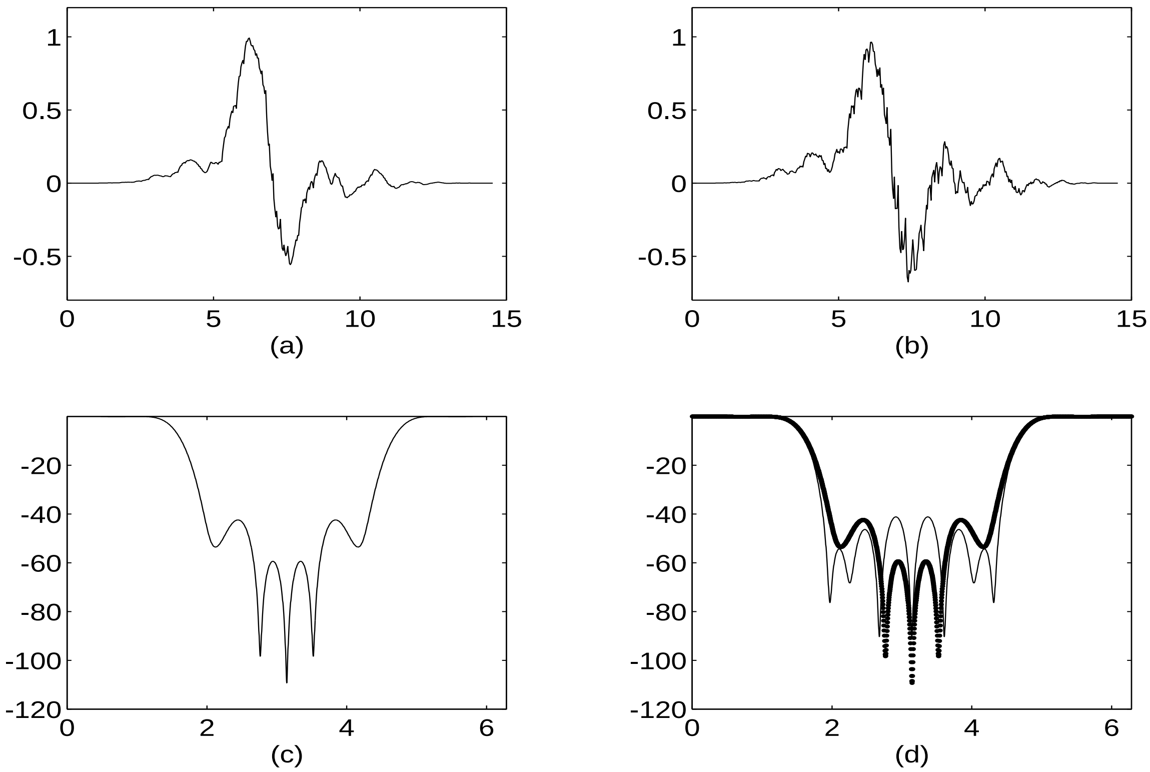

Figure 11: TV Example: (a) and (c) show $b_5(n)$ and the frequency response $H_0(\omega)$ respectively for our novel design (minimization of $\mathcal{TV}_5\mathcal{E}$). (b) and (d) show $b_5(n)$ and the frequency response $H_0(\omega)$ respectively for a standard design (minimization of $\mathcal{E}$.)

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