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Lemma 1 (Koilpillai) *The filters $H_k(z)$ in a CM filter bank obtained from a real, even-symmetric prototype filter $G(z)$ form a unitary filter bank iff*

$$G_n(z^{-1})G_n(z)+G_{M+n}(z^{-1})G_{M+n}(z) = \frac{2}{M},\ 0 \le n \le M-1 \tag{8}$$

Let $J = \lfloor \frac{M}{2} \rfloor$. Then, unitary CM filter banks of length $N = 2mM$ obtained as above are characterized by J "power-complementary" pairs. This follows from the even-symmetry of $g(n)$ which in turn ensures that Eqn. 8 is the same for k and $M-1-k$. For odd M, G_J is verified to be a pure delay. Now each of the power-complementary pairs are completely parameterized as follows: $\begin{bmatrix} G_n \\ G_{n+M} \end{bmatrix} =$

$$\frac{1}{\sqrt{2M}} \prod_{l=m-1}^{1} \begin{bmatrix} \cos(\theta_{n,l}) & z^{-1}\sin(\theta_{n,l}) \\ \sin(\theta_{n,l}) & -z^{-1}\cos(\theta_{n,l}) \end{bmatrix} \begin{bmatrix} \cos(\theta_{n,0}) \\ \sin(\theta_{n,0}) \end{bmatrix} \tag{9}$$

Thus precisely Jm parameters (roughly $N/4$) determine a unitary CM filter bank. We have a CMTF from this filter bank iff

$$\sum_{k=0}^{N-1} h_0(k) = \sum_{n=0}^{2M-1} c_{0,n}G_n(-1) = \sqrt{M}. \tag{10}$$

One expects this condition, to translate into one constraint on the θ's. However, it turns out that it translates into J constraints, and moreover we get the nice result (for proof see Gopinath [5]):

Lemma 2 *Let G_n be as above and let $\Theta_n = \sum_{j=0}^{m-1} \theta_{n,j}$. Then Eqn. 10 is satisfied iff*

$$\Theta_n = \frac{\pi}{2}m - \frac{\pi}{4M}(2n+1) \tag{11}$$

and therefore we have

Theorem 1 *For every integer M, there exists CM wavelet tight frames of length $2Mm$, that can be explicitly parameterized by $J(m-1)$ parameters.*

Notice that there is an independent constraint on each of the J "power-complementary" pairs. Thus we have an explicitly parameterization of these CMTFs.

2.1. Orthonormality

Multiplicity M wavelet tight frames are ON iff the scaling function and its integer translates form an ON family. This in turn is the case iff a certain matrix, corresponding to the scaling vector (called the Lawton matrix), has a unique eigenvector of eigenvalue 1. It is always the case that the Lawton matrix has an eigenvector of eigenvalue 1. Thus non-orthonormality is equivalent to having an additional eigenvector with eigenvalue 1. Since this is an algebraic constraint, for N finite, the set of all non-orthonormal wavelet tight frames (for fixed N) forms an algebraic sub-variety of the set of all wavelet tight frames corresponding to N. Therefore, almost all wavelet tight frames are ON wavelet bases. In the case of CM wavelet tight frames, we have *not* been able to construct a single non-orthonormal tight frame. Though this is inconclusive, we strongly believe that the following is true.

Conjecture 1 *All CM wavelet tight frames are ON wavelet bases.*

2.2. Regularity

It can easily be shown that K regularity is equivalent to $(M-1)(K-1)$ linear constraints on the scaling vector. These in turn become non-linear constraints on the θ's that parameterize a CM tight frame. Using numerical techniques we have been able to solve and obtain examples of regular CM wavelet bases.

Conjecture 2 *There exist K regular multiplicity M CM wavelet bases for all M and K.*

3. OPTIMAL DESIGN OF CMTFS

Given a signal or classes of signals, the problem of design of optimal scaling functions, that best represents the signal(s) has been solved [6, 11]. Using the same theory we design *CM scaling functions* that optimally represent signal(s). Notice that this becomes a design of the entire wavelet basis (not just the scaling function). Also, the number of parameters in the optimization is much less ($N\lfloor \frac{M}{2} \rfloor / 2M$ as opposed to $(M-1)(N/M-1)$).

4. SMOOTH CMTFS

Regular CM wavelet bases require long h_0 ($N \approx 4MK$) because of the additional constraints. In contrast, general K regular tight frames exist with $N = MK$. Moreover, regularity is a property of h_0, and does not necessarily imply smoothness of ψ_0. Hence one wishes/expects to obtain smooth ψ_0 without imposing large regularity K. Intuition suggests that for smooth signals, the optimal scaling function (designed as above) would be smooth. Using this idea we generate smooth CM wavelet bases.

5. CONCLUSION

In this paper compactly supported, CM, multiplicity M wavelet tight frames are constructed. For such bases, the scaling function uniquely determines the wavelets. Solving a system of non-linear equations, we also construct K-regular CM wavelets. We conjecture that all CM tight frames are ON bases. We show how to design optimal wavelets for a given signal (or classes of signals) and use this to generate smooth wavelets (without imposing regularity on the scaling vector). Recently odd-length CM filter banks, with no symmetry restrictions on $g(n)$ have been designed [10]. One expects corresponding CMTFs and this issue is currently being investigated.

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COSINE-MODULATED ORTHONORMAL WAVELET BASES

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ABSTRACT

To give flexibility to the time-frequency resolution tradeoff of orthonormal (ON) wavelet bases constructed by I. Daubechies [2, 1], recently multiplicity M ON wavelet bases (more generally tight frames (TFs)) have been constructed by several authors [3, 4, ?, 7, 12]. These generalizations of the multiplicity 2 ON wavelet bases of I. Daubechies, differ from the latter in that, while the multiresolution analysis is uniquely determined by the scaling function $\psi_0(t)$, the wavelets are not. In this paper we give a parameterization of even-length multiplicity M cosine-modulated wavelet tight frames (CMTFs), where both the wavelets and the multiresolution analysis are uniquely determined by $\psi_0(t)$. We accomplish this using the recently developed theory of cosine-modulated (CM) filter banks [8, ?, 9]. We also construct "regular" CMTFs by a solving a set of non-linear equations. We conjecture that for all regularities CMTFs exist. All the TFs constructed turn out to be orthonormal bases. We conjecture that this is always the case. We then design optimal CMTFs for the representation of a given signal (or classes of signals). Since regular CMTFs tend to have long support and have to be designed numerically, and since regularity is not the same as smoothness of ψ_0 (which one desires), we design smooth CMTFs by designing optimal CMTFs for arbitrary smooth functions (based on the intuition that the optimal wavelet for a smooth function will probably be smooth).

1. INTRODUCTION

Given a multiplicity M *scaling vector* h_0 of length N, there exist $(M-1)$ wavelet vectors h_l also of length N, a unique scaling function $\psi_0(t)$ supported in $[0,(N-1)/(M-1)]$ and $(M-1)$ wavelets $\psi_l(t)$, also supported in $[0,(N-1)/(M-1)]$, that satisfy $\sum_{k=0}^{N-1} h_l(k) = \sqrt{M}\delta(l)$,

$$\sum_{k=0}^{N-1} h_l(k)h_m(k+Mn) = \delta(n)\delta(l-m), \tag{1}$$

and

$$\psi_l(t) = \sqrt{M}\sum_{k=0}^{N-1} h_l(k)\psi_0(Mt-k). \tag{2}$$

If we define

$$\psi_{l,j,k} = M^{j/2}\psi_l(M^j t - k) \tag{3}$$

then $\{\psi_{l,j,k}\}$ forms a TF for $L^2(\mathbb{R})$, i.e $\forall f \in L^2(\mathbb{R})$,

$$f(t) = \sum_{l=1}^{M-1}\sum_{j,k} \langle f, \psi_{l,j,k}(t)\rangle\, \psi_{i,j,k}. \tag{4}$$

A scaling vector is said to be K regular if

$$H_0(z) = (1 + z^{-1} + \ldots + z^{-(M-1)})^K P(z) \tag{5}$$

for maximal K and FIR $P(z)$. Associated TFs are said to be K-regular. Thus, given h_0 one can construct a bevy of TFs. There are two ways to construct h_0 developed earlier [4, 3]. The first method is based on a parameterization of all h_0 of length N with $(M-1)(N/M-1)$ parameters [4]. However, h_0 may not be regular. The second method generates regular h_0 by solving a certain set of linear equations [3]. In both methods, h_0 uniquely determines ψ_0 (multiresolution analysis). However, $\{h_l\}$ and $\{\psi_l\}$ are non-unique and parameterized by precisely $\binom{M-1}{2}$ free parameters.

2. CMTFS

We now give a third approach, the CM approach, to generating h_0. Here we demand that h_0 and h_l are of the form

$$h_l(n) = g(n)\cos(\frac{\pi}{2M}(2l+1)(n-\frac{N-1}{2}) + \frac{\pi}{4}) = g(n)c_{l,n} \tag{6}$$

for an even-symmetric sequence $g(n)$ of length $N = 2Mm$ and $l = 0,1,\ldots,M-1$. Notice that (Eqn. 1) this ensures that the wavelets are also uniquely determined by h_0. Moreover, from the cosine-modulation property, the wavelets are essentially bandpass functions occupying adjacent frequency bins.

Scaling and wavelet vectors give rise to the analysis filters of a unitary filter bank [4]. Moreover, given a unitary filter bank, if the lowpass filter evaluates to $\sqrt{M}$ at 0 frequency then the filters act as scaling and wavelet vectors, and hence we have a tight frame. If we write $G(z)$ in its polyphase representation, $G(z) = \sum_{n=0}^{2M-1} z^{-n}G_n(z^{2M})$, then,

$$H_l(z) = \sum_{n=0}^{2M-1} c_{l,n}z^{-n}G_n(-z^{2M}). \tag{7}$$

The conditions on $g(n)$ to obtain a unitary filter bank can now be obtained as follows [8].

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