Equivalently,

$$\sum_k h_i'(k) g_j'(Ml-k) = \delta(i-j)\delta(l), \tag{45}$$

and the new filter bank is PR. When $(N,M) \neq 1$, the above relationship does not hold, and therefore, in general, the new filter bank is not PR. □

Remark: Even if $(N,M) \neq 1$, the new filter bank *may be* PR. For example, if $H_p(z) = I$, then, independent of N, the interpolated filter bank is PR.

Theorem 1 *Let $\{\psi_{i,j,k}\}$ and $\left\{\tilde{\psi}_{i,j,k}\right\}$ form a frame, dual frame pair of an FB wavelet frame. Define,*

$$\psi_i'(t) = \frac{1}{\sqrt{N}}\psi_i\left(\frac{t}{N}\right) \; ; \; \tilde{\psi}_i'(t) = \frac{1}{\sqrt{N}}\tilde{\psi}_i\left(\frac{t}{N}\right) \tag{46}$$

$$\psi_{i,j,k}' = M^{j/2}\psi_i'(M^j t - k) \; ; \; \tilde{\psi}_{i,j,k}' = M^{j/2}\tilde{\psi}_i'(M^j t - k). \tag{47}$$

Then, $\{\psi_{i,j,k}'\}$ and $\left\{\tilde{\psi}_{i,j,k}'\right\}$ form a frame, dual frame pair for $L^2(\mathbb{R})$, if $(N,M) = 1$. In general if $(N,M) \neq 1$, the pair does not form a frame, dual frame pair.

Proof: From Lemma 3, the functions $\psi_i'(t)$ and $\tilde{\psi}_i'(t)$ are generated by replacing the filters $H_i(z)$ by $H'(z) = H_i(z^N)$ and $G_i(z)$ by $G_i'(z) = G_i(z^N)$ respectively. Clearly, these functions form a wavelet frame iff the corresponding filters form a PR filter bank. From Lemma 4, this occurs iff $(N,M) = 1$. Hence the result follows. □

Theorem 2 *Let $\{\psi_{i,j,k}\}$ form an FB wavelet frame for $L^2(\mathbb{R})$ with dual frame $\left\{\tilde{\psi}_{i,j,k}\right\}$. Then $\left\{\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{N}}\right\}$ and $\left\{\frac{1}{\sqrt{N}}\tilde{\psi}_{i,j,\frac{k}{N}}\right\}$ form dual frames for $L^2(\mathbb{R})$ if $(N,M) = 1$. In general, if $(N,M) \neq 1$, they do not form a frame, dual frame pair for $L^2(\mathbb{R})$.*

Proof:

$$\begin{aligned}
\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{\sqrt{N}}}(t) &= \frac{1}{\sqrt{N}}\left[M^{j/2}\psi_i\left(M^j t - \frac{k}{N}\right)\right] \\
&= \frac{1}{\sqrt{N}}\left[M^{j/2}\psi_i\left(\frac{M^j}{N}(Nt - M^{-j}k)\right)\right] \\
&= M^{j/2}\psi_i'\left(M^j(Nt - M^{-j}k)\right) \\
&= M^{j/2}\psi_i'\left(M^j Nt - k)\right) \\
&= \psi_{i,j,k}'(Nt)
\end{aligned} \tag{48}$$

Therefore $\left\{\frac{1}{\sqrt{N}}\psi_{i,j,\frac{k}{N}}\right\}$ and $\left\{\frac{1}{\sqrt{N}}\tilde{\psi}_{i,j,\frac{k}{N}}\right\}$ form a frame, dual frame pair, iff $\{\psi_{i,j,k}'(Nt)\}$ and $\left\{\tilde{\psi}_{i,j,k}'(Nt)\right\}$ form a frame, dual frame pair for $L^2(\mathbb{R})$. From Theorem 1 the result follows. □

5. CONCLUSION

In this paper we show that for all wavelet frames that have filter banks associated with them, oversampling by a factor or N preserves the frame property with the same dual frame (oversampled by N) iff $(N,M) = 1$. In particular, tight frames are preserved by oversampling by N, with $(N,M) = 1$. Our results are sharper than the results in [1] for the case of FB wavelet frames. We conjecture that all compactly supported wavelet frames are FB wavelet frames.

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and the sequence g_0 is the *dual scaling vector*. The remaining filters h_i are the *wavelet vectors*, and the corresponding g_i constitute the *dual wavelet vectors*. The functions $\psi_0(t)$ and $\tilde{\psi}_0(t)$ are the scaling function and dual scaling function respectively. Given that these functions exist, the wavelets (and dual wavelets) are defined by

$$\psi_i(t) = \sqrt{M} \sum_k h_i(k) \psi_0(Mt - k) \tag{29}$$

$$\tilde{\psi}_i(t) = \sqrt{M} \sum_k g_i(k) \psi_0(Mt + k) \tag{30}$$

From the (algebraic) PR conditions, it can be shown that $\{\psi_{i,j,k}\}$ and $\left\{\tilde{\psi}_{i,j,k}\right\}$ form dual wavelet frames [6]. In the Fourier transform domain the scaling functions and wavelets are given by

$$\hat{\psi}_i(\omega) = \left[\frac{1}{\sqrt{M}} H_i(e^{i\omega/M})\right] \prod_{j=2}^{\infty} \left[\frac{1}{\sqrt{M}} H_0(e^{i\omega/M^j})\right] \tag{31}$$

$$\hat{\tilde{\psi}}_i(\omega) = \left[\frac{1}{\sqrt{M}} G_i(e^{-i\omega/M})\right] \prod_{j=2}^{\infty} \left[\frac{1}{\sqrt{M}} G_0(e^{-i\omega/M^j})\right] \tag{32}$$

We have seen how a filter can give rise to a frame. Such frames will be called *FB wavelet frames*. Are all wavelet frames FB wavelet frames? Given a wavelet frame, to obtain a filter bank from it, it suffices to obtain the scaling and dual scaling functions since, the filter bank can then be obtained from,

$$H_i(e^{i\omega}) = \hat{\psi}_0(M\omega)/\hat{\psi}_0(\omega) \tag{33}$$

and

$$G_i(e^{-i\omega}) = \hat{\tilde{\psi}}_0(M\omega)/\hat{\tilde{\psi}}_0(\omega) \tag{34}$$

Moreover, if the frame is also a biorthogonal basis, then, the filter bank can also be obtained as follows:

$$g_i(k) = \sqrt{M} \int_{\mathbb{R}} \psi_i(t) \tilde{\psi}_0(Mt + k) \tag{35}$$

and

$$h_i(k) = \sqrt{M} \int_{\mathbb{R}} \tilde{\psi}_i(t) \psi_0(Mt - k) \tag{36}$$

Constructing a filter bank associated with a wavelet frame is equivalent to constructing the scaling function. J. L. Journé has exhibited ([9]) an infinitely supported (with poor decay), multiplicity 2, orthonormal wavelet basis for which there *does not* exist a scaling function, thus proving that not all wavelet frames are FB wavelet frames. However, we believe that every compactly supported, multiplicity M, wavelet frame is an FB wavelet frame. Consider a wavelet tight frame. One can define,

$$\left|\hat{\psi}_0(\omega)\right|^2 = \sum_{i=1}^{M-1} \sum_{j=1}^{\infty} \left|\hat{\psi}_i(\omega/M^j)\right|^2 \tag{37}$$

and

$$|H_i(e^{i\omega})|^2 = \left|\hat{\psi}_i(M\omega)\right|^2 / \left|\hat{\psi}_0(\omega)\right|^2 \tag{38}$$

Now one of the spectral factors of the left hand side of the above equation for $i = 0$, will be $H_0(e^{i\omega})$, such that $\psi_0(t)$ defined by Eqn. 27 will satisfy Eqn. 37. Therefore, if Eqn. 37 and Eqn. 38 are well defined, and we find the right spectral factor, then the wavelet tight frame is associated with a filter bank. In the frame situation also we can find corresponding equations, and therefore we make the conjecture:

Conjecture 1 *Every multiplicity M compactly supported wavelet frame is an FB wavelet frame.*

4. THE MAIN RESULTS

We need a set of preliminary results.

Lemma 2 *Given $H(z)$, let*

$$\hat{\psi}(\omega) \stackrel{\text{def}}{=} \prod_{j=1}^{\infty} H(e^{i\omega/M^j}) \tag{39}$$

exist as the Fourier transform of a function $\psi(t) \in L^2(\mathbb{R})$. For any positive N, If $H'(z) = H(z^N)$, for positive integer N, then,

$$\hat{\psi}'(\omega) \stackrel{\text{def}}{=} \prod_{j=1}^{\infty} H'(e^{i\omega/M^j}) = \psi(N\omega) \tag{40}$$

is the Fourier transform of $\psi'(t) \in L^2(\mathbb{R})$, given by

$$\psi'(t) = \frac{1}{N} \psi(\frac{t}{N}). \tag{41}$$

Proof: Follows by substituting $H'(e^{i\omega}) = H(e^{iN\omega})$. □

If a function is generated as in Eqn. 39, then stretching it by N, is equivalent to upsampling the sequence by N.

A direct consequence is the following result for FB wavelet frames:

Lemma 3 *Given an FB wavelet frame, if the filters are interpolated by a positive integer N, then the corresponding scaling functions and wavelets are stretched by a factor of N, and scaled by $1/N$. Conversely, if the functions are stretched by N, the filters are interpolated by N.*

Proof: Apply Lemma 2 for all the functions involved. □

Lemma 4 *If the filters of a PR filter bank are interpolated by a positive integer N, such that $(N, M) = 1$, then the resultant filter bank is also PR. Conversely, if $(N, M) \neq 1$, then, in general, the interpolated filter bank is not PR.*

Proof: For a PR filter bank $G_p^T(z) H_p(z) = I$ and therefore

$$\sum_k H_{i,k}(z) G_{j,k}(z) = \delta(i - j) \tag{42}$$

Let $h_i'(n)$ and $g_i'(n)$ denote the interpolated filters. Define $\alpha(l) = \sum_k h_i'(k) g_j'(Ml - k)$. Then from Eqn. 8,

$$\begin{aligned} \mathcal{Z}(a(l)) &= [\downarrow M] H_i'(z) G_j'(z) \\ &= [\downarrow M] \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} z^{-Nk} H_{i,k}(z^{NM}) z^{Nl} G_{j,l}(z^{NM}) \\ &= [\downarrow M] \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} z^{N(l-k)} H_{i,k}(z^{NM}) G_{j,l}(z^{NM}) \end{aligned} \tag{43}$$

When $(N, M) = 1$, as l and k range from 0 through $(M-1)$ the only value of $N(k - l)$ that is a multiple of M occurs when $k = l$ (and is 0) and therefore from Eqn. 42

$$\mathcal{Z}(a(l)) = \sum_{k=0}^{M-1} H_{i,k}(z^N) G_{j,k}(z^N) = \delta(i - j). \tag{44}$$

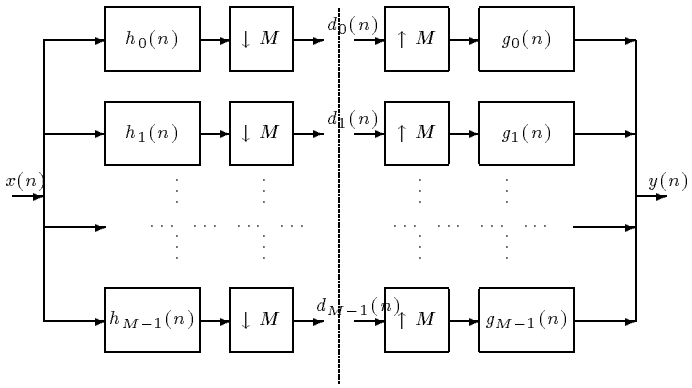

Figure 1. An M Channel Filter Bank

$$H_i(z) = \sum_{k=0}^{M-1} z^{-k} H_{i,k}(z^M) \; ; \; G_i(z) = \sum_{k=0}^{M-1} z^k G_{i,k}(z^M) \quad (8)$$

where

$$H_{i,k}(z) = \sum_n h_i(Mn+k) z^{-n} \quad (9)$$

$$G_{i,k}(z) = \sum_n g_i(Mn-k) z^{-n}. \quad (10)$$

Now from Fig. 3., we have

$$\begin{aligned} D_i(z) &= [\downarrow M] H_i(z) X(z) \\ &= [\downarrow M] \left[\sum_{k=0}^{M-1} z^{-k} H_{i,k}(z^M) \right] \left[\sum_{j=0}^{M-1} z^j X_j(z^M) \right] \\ &= \sum_{k=0}^{M-1} H_{i,k} X_k. \end{aligned} \quad (11)$$

and similarly, for the synthesis bank,

$$\begin{aligned} \sum_i z^k Y_k(z^M) &= \sum_i \{[\uparrow M] D_i(z)\} G_i(z) \\ &= \sum_{i,k} z^k D_i(z^M) G_{i,k}(z^M) \\ &= \sum_k z^k \left[\sum_i D_i(z^M) G_{i,k}(z^M) \right]. \end{aligned} \quad (12)$$

and therefore

$$Y_k(z) = \sum_i G_{i,k}(z) D_i(z). \quad (13)$$

In terms of the *polyphase* component matrices defined by,

$$(H_p(z))_{i,j} = H_{i,j}(z) \quad (14)$$

$$(G_p(z))_{i,j} = G_{i,j}(z), \quad (15)$$

we have with

$$\mathbf{x}_p(z) = [X_0(z), X_1(z), \ldots, X_{M-1}(z)]^T$$

$$\mathbf{y}_p(z) = [Y_0(z), Y_1(z), \ldots, Y_{M-1}(z)]^T$$

$$\mathbf{d}(z) = [D_0(z), D_1(z), \ldots, D_{M-1}(z)]^T,$$

$$\mathbf{d}(z) = H_p(z) \mathbf{x}_p(z) \quad (16)$$

$$\mathbf{y}_p(z) = G_p^T(z) \mathbf{d}(z) \quad (17)$$

$$\mathbf{y}_p(z) = G_p^T(z) H_p(z) \mathbf{x}_p(z). \quad (18)$$

Perfect reconstruction (PR) means $\mathbf{x}_p(z) = \mathbf{y}_p(z)$. Therefore, we have the following characterization of PR filter banks (for a proof see [6]):

Lemma 1 *A filter bank is PR iff one of the following equivalent statements is true:*

1.
$$G_p^T(z) H_p(z) = H_p^T(z) G_p(z) = I \quad (19)$$

2.
$$\sum_k h_i(k) g_j(Ml-k) = \delta(l)\delta(i-j) \quad (20)$$

3.
$$\sum_i \sum_k h_i(Mk+n_1) g_i(-Mk-n_2) = \delta(n_1-n_2) \quad (21)$$

A filter bank is *unitary* $H_p(z)$ is unitary on the unit circle.

$$H_p^T(z^{-1}) H_p(z) = I. \quad (22)$$

For unitary filter banks

$$G_p(z) = H_p(z^{-1}) \quad (23)$$

and therefore,

$$G_i(z) = H_i(z^{-1}) \quad (24)$$

The unitariness (or PR) condition for unitary filter banks can also be expressed as

$$\sum_k h_i(k) h_j(k+Ml) = \delta(l)\delta(i-j) \quad (25)$$

$$\sum_i \sum_l h_i(Ml+n_1) h_i(Ml+n_2) = \delta(n_1-n_2) \quad (26)$$

Unitary filter banks give rise to wavelet tight frames just as general filter banks give rise to frames. The PR conditions ensure that filter banks give a natural change of coordinates in separable Hilbert spaces [6, 4]. It turns out that the filters can be used to construct an wavelet frame under some additional analytical conditions than essentially ensure that all the wavelets exist as functions in $L^2(\mathbb{R})$. For details of this construction see [6]. We now give the salient features in the methodology of going from a filter bank to a multiplicity M wavelet frame.

Given a filter bank define the following functions:

$$\hat{\psi}_0(\omega) = \prod_{j=1}^{\infty} \left[\frac{1}{\sqrt{M}} H_0(e^{\imath\omega/M^j}) \right] \quad (27)$$

$$\hat{\tilde{\psi}}_0(\omega) = \prod_{j=1}^{\infty} \left[\frac{1}{\sqrt{M}} G_0(e^{-\imath\omega/M^j}) \right] \quad (28)$$

and assume that they are Fourier transforms of functions in $L^2(\mathbb{R})$. In order to ensure this the sequences must indeed satisfy some technical conditions, irrelevant to our discussion. For the existence of a tight frame, however, it suffices to have $H_0(1) = \sqrt{M}$. The sequence h_0 is the *scaling vector*,

OVERSAMPLING INVARIANCE OF WAVELET FRAMES

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1. INTRODUCTION

Wavelet theory has spawned a renewed interest in frame theory [13, 1]. Wavelet theory (of multiplicity 2) studies wavelet frames - sets of functions $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$, generated from a single function $\psi(t)$. In their paper, Chui and Shi, [1], raise and solve the problem of oversampling of such a wavelet frame. They work with frame bounds and show that, given a wavelet frame generated by ψ, $\left\{N^{-1/2}\psi_{j,k/N}\right\}$ is also a wavelet frame for all odd N. In this paper, we use filter bank theory, (which gives the most versatile technique for the construction of wavelet frames of arbitrary multiplicity M, and is a source of efficient computational algorithms [6, 5, 3, 2, 8, 7]) to show that for wavelet frames associated with filter banks, oversampling preserves frames iff (notice that our result is sharper), in the generic case, $(M, N) = 1$. Since we use filter bank theory, the result is true for all frames that have filter banks associated with them. This is the case for all frames, where an associated *scaling function* can be constructed. We conjecture that for every compactly supported frame, has an associated *scaling function*.

2. FRAMES

A set of $(M-1)$ functions (wavelets) $\psi_i(t)$, $i = 1, \ldots, M-1$, generates multiplicity M wavelet frame if $\{\psi_{i,j,k}\}$ defined by

$$\psi_{i,j,k} = M^{j/2}\psi_i(M^j t - k) \tag{1}$$

satisfy

$$A\,\|f\|^2 \le \sum_{i,j,k} |\langle f, \psi_{i,j,k}\rangle|^2 \le B\,\|f\|^2 \tag{2}$$

for all $f \in L^2(\mathbb{R})$ and for $B \ge A > 0$. The numbers A and B are called the frame bounds.

For every frame, there exists a *dual* frame with frame bounds frame with frame bounds $1/A \ge 1/B > 0$ ([13]). Therefore, given a wavelet frame, there exist $(M-1)$ functions (dual wavelets) $\tilde{\psi}_i(t)$, such that if

$$\tilde{\psi}_{i,j,k} = M^{j/2}\tilde{\psi}_i(M^j t - k), \tag{3}$$

$$\frac{1}{B}\,\|f\|^2 \le \sum_{i,j,k} |\langle f, \psi_{i,j,k}\rangle|^2 \le \frac{1}{A}\,\|f\|^2 \tag{4}$$

and for all $f \in L^2(\mathbb{R})$,

$$f(t) = C\sum_{i,j,k} \langle f, \psi_{i,j,k}\rangle\,\tilde{\psi}_{i,j,k} = C\sum_{i,j,k} \left\langle f, \tilde{\psi}_{i,j,k}\right\rangle \psi_{i,j,k} \tag{5}$$

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If the frame bounds are equal, the frame is a *tight frame*. Tight frames are self dual, and hence *act* like orthonormal bases and we have for all $f \in L^2(\mathbb{R})$,

$$f(t) = \frac{1}{A}\sum_{i,j,k} \langle f, \psi_{i,j,k}\rangle\,\psi_{i,j,k} \tag{6}$$

A general approach to the construction of multiplicity M wavelet frames is given in [6]. Recently, a class of modulated wavelet tight frames has also been constructed [7, 4]. Both constructions use the perfect reconstruction filter bank theory [11, 12, 10]. with additional technical that ensure that the wavelets exist. Historically, Lawton [8] constructed multiplicity 2 tight frames, and then Cohen [2] used a general two-channel filter bank, with some restrictions to construct multiplicity 2 frames. The authors generalized these ideas to the case of multiplicity M frames and tight frames. It turns out that the theory of filter banks also gives general decomposition and recomposition lemmas for arbitrary separable Hilbert spaces [6, 4].

3. FB WAVELET FRAMES

This section briefly describes the generation of wavelet frames from filter banks. Filter banks are efficient convolution structures that have been extensively used in sub-band coders for speech, and in transmultiplexers (devices to convert time-division multiplexed data to frequency-division multiplexed data and vice-versa). In a filter bank, a data sequence $x(n)$ (see Fig. 3.), is decomposed into M channels by convolution with sequences $h_i(n), i = 1, 2, \ldots, M-1$, called the *analysis* filters, downsampled by M on each channel, upsampled by M on each channel, and then convolved with the sequences $g_i(n), i = 1, 2, \ldots, M-1$, called the *synthesis* filters, and recombined to give $y(n)$. A potential application is in speech coding, where the sequences $d_i(n)$ are encoded with different number of bits depending on the sensitivity of the ear to channel i. A filter bank is said to be a perfect reconstruction filter bank if $y(n) = x(n)$. Most of the literature in filter bank theory specializes to FIR $h_i(n)$ and $g_i(n)$ partly because FIR filters can be easily implemented, and stability issues do not arise. FIR filter bank theory includes all compactly supported wavelet frames and tight frames as special cases.

Let $X(z), Y(z)$, $H_i(z)$, and $G_i(z)$ denote the $\mathcal{Z}$-transform of the corresponding sequences (possibly non-causal and irrational). Then they can be rewritten as

$$X(z) = \sum_{k=0}^{M-1} z^k X_k(z^M)\ ;\ Y(z) = \sum_{k=0}^{M-1} z^k Y_k(z^M) \tag{7}$$

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